

# Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{g}})$ and Quantum $Z$ -algebras

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## Abstract

A new definition of the elliptic algebra  $U_{q,p}(\widehat{\mathfrak{g}})$  associated with an untwisted affine Lie algebra  $\widehat{\mathfrak{g}}$  is given as a topological algebra over the ring of formal power series in  $p$ . We also introduce a quantum dynamical analogue of Lepowsky-Wilson's  $Z$ -algebras. The  $Z$ -algebra governs the irreducibility of the infinite dimensional  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. Some level-1 examples indicate a direct connection of the irreducible  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules to those of the  $W$ -algebras associated with the coset  $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}} \supset (\widehat{\mathfrak{g}})_{diag}$  with level  $(r - g - 1, 1)$  ( $g$ : the dual Coxeter number), which includes Fateev-Lukyanov's  $WB_l$ -algebra.

## 1 Introduction

The algebra  $U_{q,p}(\widehat{\mathfrak{g}})$  is an elliptic analogue [1, 2] of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  in the Drinfeld realization [3]. There are two types of the elliptic quantum groups, the vertex type and the face type [4, 5]. Deriving the  $L$ -operators [2, 6, 7] and introducing the Hopf-algebroid structure [8–10]  $U_{q,p}(\widehat{\mathfrak{g}})$  is now recognized as a face type elliptic quantum group.

Originally  $U_{q,p}(\widehat{\mathfrak{g}})$  with  $p = q^{2r}$  was derived for  $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}(2, \mathbb{C})$  [1] as a deformation of the screening currents of the coset conformal field theory (CFT)  $\widehat{\mathfrak{sl}}_2 \oplus \widehat{\mathfrak{sl}}_2 \supset (\widehat{\mathfrak{sl}}_2)_{diag}$  with level  $(r - k - 2, k)$  [11–16] instead of considering a deformation of  $U_q(\widehat{\mathfrak{sl}}_2)$  itself. Such coset CFT is known to be realized in terms of the level- $k$  free boson and the  $\mathbb{Z}_k$ -parafermion [17], or the  $Z$ -algebra [18] associated with the level- $k$  standard representation of  $\widehat{\mathfrak{sl}}_2^4$ . It was then crucial in [1] to realize that the level- $k$  boson should be deformed both  $q$ - and elliptically [19] whereas the  $\mathbb{Z}_k$ -parafermion gets only a  $q$ -deformation to obtain consistent relations for the generators in  $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ .

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<sup>4</sup>The difference between  $Z$ -algebra and Parafermion is whether one adds zero-modes of the bosons to it or not.

In [2], a realization of  $U_{q,p}(\widehat{\mathfrak{g}})$  for general untwisted affine Lie algebra  $\widehat{\mathfrak{g}}$  was given by modifying the Drinfeld realization of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$ . However its structure associated with the quantum  $Z$ -algebras has not yet been discussed so far. The purpose of this paper is to address this subject. The general theory of the  $Z$ -algebra was studied by Lepowsky and Wilson [18] and by Gepner [20] in the representation theory of affine Lie algebras and in CFT, respectively. Its quantum deformation and application to the representations of  $U_q(\widehat{\mathfrak{g}})$  was partially investigated in [1, 21–24]. A construction of the coset CFT associated with the general  $\widehat{\mathfrak{g}}$  was also given [25] in terms of the generalized parafermions. We extend these studies to the elliptic algebras  $U_{q,p}(\widehat{\mathfrak{g}})$ . In particular, we define a dynamical analogue  $\mathcal{Z}_k$  of the quantum  $Z$ -algebras and show that the level- $k$  highest weight representations of  $U_{q,p}(\widehat{\mathfrak{g}})$  are realized in terms of  $\mathcal{Z}_k$  and the level- $k$  elliptic bosons. It is then shown that the irreducibility of the infinite dimensional  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules is governed by the  $\mathcal{Z}_k$ -modules as in the affine Lie algebra cases [18].

On the other hand, it was conjectured [1, 2] that the  $U_{q,p}(\widehat{\mathfrak{g}})$  provides an algebra of the screening currents of the deformation of the  $W$ -algebras associated with the coset  $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}} \supset (\widehat{\mathfrak{g}})_{diag}$  with level  $(r - g - 1, 1)$ . For the simply-laced  $\widehat{\mathfrak{g}}$ , such deformed  $W$ -algebras have been realized in [26–28], and in particular for the  $\widehat{\mathfrak{sl}}_N$  case the conjecture has been established by an explicit comparison of the free field realizations [7, 26, 27, 29]. However for the non-simply laced  $\widehat{\mathfrak{g}}$ , deformation of the coset type  $W$ -algebras has not yet been studied at all. One should note that the coset type  $W$ -algebras associated with the non-simply laced  $\widehat{\mathfrak{g}}$  are different from those obtained by the quantum Hamiltonian reduction. See for example [30]. We investigate this issue further by giving an explicit realization of the level-1 irreducible highest weight representations of  $U_{q,p}(\widehat{\mathfrak{g}})$  for  $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ . We show that at least for  $A_l^{(1)}$  and  $D_l^{(1)}$  the level-1 elliptic currents  $e_j(z)$  and  $f_j(z)$  coincide with the screening currents of the deformed  $W$ -algebras obtained in [26–28]. We also show that the irreducible representations of  $U_{q,p}(\widehat{\mathfrak{g}})$  is naturally decomposed into a direct sum of the irreducible  $W$ -algebras of the coset type for  $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}$ . This suggests in particular an existence of a deformation of Fateev-Lukyanov’s  $WB_l$ -algebra [31] as the commutant of the screening operators provided by the level-1 elliptic currents  $e_j(z)$  and  $f_j(z)$  of  $U_{q,p}(B_l^{(1)})$ .

It is also worth to mention that the coset type  $W$ -algebras describe a critical behavior of the face type elliptic solvable lattice models [32, 33]. Correspondingly the  $U_{q,p}(\widehat{\mathfrak{g}})$  provides an algebraic framework to formulate the lattice model itself in the spirit of Jimbo and Miwa [34]. This has been established for  $\widehat{\mathfrak{sl}}_N$  in [1, 2, 7, 10, 35, 36] by constructing the  $L$ -operator and introducing the Hopf algebroid structure. In order to construct the  $L$ -operator of  $U_{q,p}(\widehat{\mathfrak{g}})$  and also to get a realization of a generating function of the deformation of the  $W$ -algebras, it is crucial to introduce new types of elliptic bosons, which we call the fundamental weight type  $A_m^j$  and the orthonormal basis type  $\mathcal{E}_m^{\pm j}$  distinguishing from the usual ones  $\alpha_{j,m}$  ( $\alpha_{j,m}^\vee$ ) corresponding to

the simple (co-)root and appearing as generators of  $U_{q,p}(\widehat{\mathfrak{g}})$ . An idea of such bosons has already appeared in [26–28]. We give an explicit construction of them for  $\widehat{\mathfrak{g}} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ . As a check we calculate the commutation relations among  $\mathcal{E}_m^{\pm j}$  as well as among the elliptic currents  $k_{\pm j}(z)$ , the generating functions of  $\mathcal{E}_m^{\pm j}$ , and show that they have a universal form. See Theorem 5.3 and 5.7.

This paper is organized as follows. In section 2, we define the elliptic algebra  $U_{q,p}(\widehat{\mathfrak{g}})$  as a topological algebra generated by the elliptic Drinfeld generators. This is a new definition of  $U_{q,p}(\widehat{\mathfrak{g}})$  given independently of  $U_q(\widehat{\mathfrak{g}})$  unlike the previous one in Appendix A in [2]. In section 3, we define a quantum dynamical analogue  $\mathcal{Z}_{\mathcal{V}}$  of Lepowsky and Wilson’s  $Z$ -algebra associated with the level- $k$   $U_{q,p}(\widehat{\mathfrak{g}})$ -module  $\mathcal{V}$  and its universal counterpart  $\mathcal{Z}_k$ . The irreducibility of the level- $k$  highest weight representation of  $U_{q,p}(\widehat{\mathfrak{g}})$  is shown to be governed by the  $\mathcal{Z}_k$ -module. In section 4, we give a simple realization of  $\mathcal{Z}_k$  in terms of the quantum (non-dynamical)  $Z$ -algebra associated with the level- $k$   $U_q(\widehat{\mathfrak{g}})$ -module and define a standard representation of  $U_{q,p}(\widehat{\mathfrak{g}})$ . We provide some level-1 examples of the standard representations and discuss their relation to the deformation of the  $W$ -algebras. In section 5, we give a construction of the new elliptic bosons of the fundamental weight type and the orthonormal basis type and derive various commutation relations.

## 2 Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{g}})$

### 2.1 Definition

Let  $\widehat{\mathfrak{g}} = X_l^{(1)}$  be an untwisted affine Lie algebra associated with the generalized Cartan matrix  $A = (a_{ij})$   $i, j \in \{0\} \cup I$ ,  $I = \{1, \dots, l\}$ . We denote by  $B = (b_{ij})$ ,  $b_{ij} = d_i a_{ij}$  the symmetrization of  $A$ . We take  $d_i = 1$  ( $i \in I$ ) for the simply laced cases,  $d_i = 1$  ( $1 \leq i \leq l-1$ ),  $d_l = 1/2$  for  $B_l^{(1)}$  and  $d_i = 1$  ( $1 \leq i \leq l-1$ ),  $d_l = 2$  for  $C_l^{(1)}$ . Let  $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$  and set  $q_i = q^{d_i}$ . Let  $p$  be an indeterminate.

Let  $\mathfrak{h} = \widetilde{\mathfrak{h}} \oplus \mathbb{C}d$ ,  $\widetilde{\mathfrak{h}} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$ ,  $\bar{\mathfrak{h}} = \oplus_{i \in I} \mathbb{C}h_i$  be the Cartan subalgebra of  $\widehat{\mathfrak{g}}$ . Define  $\delta, \Lambda_0, \alpha_i$  ( $i \in I$ )  $\in \mathfrak{h}^*$  by

$$\langle \alpha_i, h_j \rangle = a_{j,i}, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \quad (2.1)$$

the other pairings are 0. We also define  $\bar{\Lambda}_i$  ( $i \in I$ )  $\in \mathfrak{h}^*$  by

$$\langle \bar{\Lambda}_i, h_j \rangle = \delta_{i,j}.$$

We set  $\bar{\mathfrak{h}}^* = \oplus_{i \in I} \mathbb{C}\bar{\Lambda}_i$ ,  $\widetilde{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$ ,  $\mathcal{Q} = \oplus_{i \in I} \mathbb{Z}\alpha_i$  and  $\mathcal{P} = \oplus_{i \in I} \mathbb{Z}\bar{\Lambda}_i$ . Let  $N = l + 1$  for  $X_l = A_l$ ,  $= l$  for  $B_l, C_l, D_l$ ,  $= 7$  for  $E_6$ ,  $= 8$  for  $E_7, E_8$ ,  $= 3$  for  $G_2$ ,  $= 4$  for  $F_4$  and consider the

orthonormal basis  $\{\xi_j \ (1 \leq j \leq N)\}$  in  $\mathbb{R}^N$  with the inner product  $(\xi_j, \xi_k) = \delta_{j,k}$ . For  $A_l$ , we also set

$$\bar{\xi}_j = \xi_j - \frac{1}{l+1} \sum_{j=1}^{l+1} \xi_j. \quad (2.2)$$

We define  $\epsilon_j = \bar{\xi}_j$  for  $A_l$  and  $= \xi_j$  for other  $X_l$ . The simple roots  $\alpha_j$  and the fundamental weights  $\bar{\Lambda}_j$  ( $1 \leq j \leq l$ ) can be expressed as a linear sum of  $\epsilon_j$  [37, 38]. We follow Kac's conventions. We define  $h_{\epsilon_j} \in \bar{\mathfrak{h}}$  ( $j \in I$ ) by  $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$  and  $h_\alpha \in \bar{\mathfrak{h}}$  for  $\alpha = \sum_j c_j \epsilon_j$ ,  $c_j \in \mathbb{C}$  by  $h_\alpha = \sum_j c_j h_{\epsilon_j}$ . We regard  $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$  as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k]. \quad (2.3)$$

In particular, we have  $[h_j, \alpha_k] = a_{jk}$ . We also set  $h^j = h_{\bar{\Lambda}_j}$ .

In order to treat the dynamical shifts in the face type elliptic algebra systematically, we introduce another Heisenberg algebra generated by  $P_\alpha$  and  $Q_\beta$  ( $\alpha, \beta \in \bar{\mathfrak{h}}^*$ ) satisfying the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}]. \quad (2.4)$$

We also set

$$[P_{\epsilon_j}, \alpha] = [Q_{\epsilon_j}, \alpha] = 0, \quad [P_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = [Q_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = 0 \quad (2.5)$$

where  $P_\alpha = \sum_j c_j P_{\epsilon_j}$  for  $\alpha = \sum_j c_j \epsilon_j$ . We set  $P_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C} P_{\epsilon_j}$ ,  $Q_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C} Q_{\epsilon_j}$ ,  $P_j = P_{\alpha_j^\vee}$ ,  $P^j = P_{\bar{\Lambda}_j}$  and  $Q_j = Q_{\alpha_j}$ ,  $Q^j = Q_{\bar{\Lambda}_j^\vee}$ . Here  $\alpha_j^\vee = 2\alpha_j/(\alpha_j, \alpha_j)$ .

For the abelian group  $\mathcal{R}_Q = \sum_{j=1}^N \mathbb{Z} Q_{\alpha_j}$ , we denote by  $\mathbb{C}[\mathcal{R}_Q]$  the group algebra over  $\mathbb{C}$  of  $\mathcal{R}_Q$ . We denote by  $e^\alpha$  the element of  $\mathbb{C}[\mathcal{R}_Q]$  corresponding to  $\alpha \in \mathcal{R}_Q$ . These  $e^\alpha$  satisfy  $e^\alpha e^\beta = e^{\alpha+\beta}$  and  $(e^\alpha)^{-1} = e^{-\alpha}$ . In particular,  $e^0 = 1$  is the identity element.

Now let us set  $H = \bar{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\epsilon_j}) + \sum_j \mathbb{C} P_{\epsilon_j} + \mathbb{C} c$  and denote its dual space by  $H^* = \bar{\mathfrak{h}}^* \oplus Q_{\bar{\mathfrak{h}}}$ . We define the pairing by (2.1),  $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$  and  $\langle Q_\alpha, h_\beta \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$ . We define  $\mathbb{F} = \mathcal{M}_{H^*}$  to be the field of meromorphic functions on  $H^*$ . We regard a function of  $P + h = \sum_j a_j (P_{\epsilon_j} + h_{\epsilon_j})$ ,  $P = \sum_j b_j P_{\epsilon_j}$  and  $c$ ,  $\hat{f} = f(P + h, P, c)$ , as an element in  $\mathbb{F}$  by  $\hat{f}(\mu) = f(\langle \mu, P + h \rangle, \langle \mu, P \rangle, \langle \mu, c \rangle)$  for  $\mu \in H^*$ .

We use the following notations.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_j = \frac{q^n - q^{-n}}{q_j - q_j^{-1}},$$

$$[n]_i! = [n]_i [n-1]_i \cdots [1]_i, \quad \left[ \begin{matrix} m \\ n \end{matrix} \right]_i = \frac{[m]_i!}{[n]_i! [m-n]_i!},$$

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad (x; q, t)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m), \quad \Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.$$

**Definition 2.1.** The elliptic algebra  $U_{q,p}(\widehat{\mathfrak{g}})$  is a topological algebra over  $\mathbb{F}[[p]]$  generated by  $\mathcal{M}_{H^*}$ ,  $e_{j,m}, f_{j,m}, \alpha_{j,n}^\vee, K_j^\pm$ , ( $j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}$ ),  $d$  and the central element  $c$ . We assume  $K_j^\pm$  are invertible and set

$$\begin{aligned} e_j(z) &= \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \\ \psi_j^+(q^{-\frac{c}{2}} z) &= K_j^+ \exp \left( -(q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1-p^n} z^n \right) \exp \left( (q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1-p^n} z^{-n} \right), \\ \psi_j^-(q^{\frac{c}{2}} z) &= K_j^- \exp \left( -(q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1-p^n} z^n \right) \exp \left( (q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1-p^n} z^{-n} \right). \end{aligned}$$

Note that  $\psi_j^\pm(z)$  are formal Laurent series in  $z$ , whose coefficients are well defined in the  $p$ -adic topology. We call  $e_j(z), f_j(z), \psi_j^\pm(z)$  the elliptic currents. The defining relations are as follows. For  $g(P), g(P+h) \in \mathcal{M}_{H^*}$ ,

$$g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.6)$$

$$g(P+h)f_j(z) = f_j(z)g(P+h - \langle \alpha_j, P+h \rangle), \quad g(P)f_j(z) = f_j(z)g(P), \quad (2.7)$$

$$[g(P), \alpha_{i,m}^\vee] = [g(P+h), \alpha_{i,m}^\vee] = 0, \quad (2.8)$$

$$g(P)K_j^\pm = K_j^\pm g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.9)$$

$$g(P+h)K_j^\pm = K_j^\pm g(P+h - \langle Q_{\alpha_j}, P \rangle), \quad (2.10)$$

$$[d, g(P)] = [d, g(P+h)] = 0, \quad (2.11)$$

$$[d, \alpha_{j,n}^\vee] = n\alpha_{j,n}^\vee, \quad [d, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [d, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z), \quad (2.12)$$

$$K_i^\pm e_j(z) = q_i^{\mp a_{ij}} e_j(z) K_i^\pm, \quad K_i^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm, \quad (2.13)$$

$$[\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{ij}m]_i [cm]_j}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.14)$$

$$[\alpha_{i,m}^\vee, e_j(z)] = \frac{[a_{ij}m]_i}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \quad (2.15)$$

$$[\alpha_{i,m}^\vee, f_j(z)] = -\frac{[a_{ij}m]_i}{m} z^m f_j(z), \quad (2.16)$$

$$z_1 \frac{(q^{b_{ij}} z_2 / z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_\infty} e_i(z_1) e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_\infty} e_j(z_2) e_i(z_1), \quad (2.17)$$

$$z_1 \frac{(q^{-b_{ij}} z_2 / z_1; p)_\infty}{(pq^{b_{ij}} z_2 / z_1; p)_\infty} f_i(z_1) f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1 / z_2; p)_\infty}{(pq^{b_{ij}} z_1 / z_2; p)_\infty} f_j(z_2) f_i(z_1), \quad (2.18)$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left( \delta(q^{-c} z_1 / z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right), \quad (2.19)$$

$$\begin{aligned} & \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)} / z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p^*)_\infty} \\ & \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_\infty} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_\infty}{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_\infty} \\ & \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0, \end{aligned} \quad (2.20)$$

$$\begin{aligned}
& \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(pq^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}}{(pq^2 z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w / z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)} / w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)} / w; p)_{\infty}} \\
& \times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(s)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}), \quad (2.21)
\end{aligned}$$

where  $p^* = pq^{-2c}$  and  $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ . We also denote by  $U'_{q,p}(\widehat{\mathfrak{g}})$  the subalgebra obtained by removing  $d$ .

We treat the relations (2.12), (2.15)-(2.21) as formal Laurent series in  $z, w$  and  $z_j$ 's. In each term of (2.17)-(2.21), the expansion direction of the structure function given by a ratio of infinite products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of (2.17),  $\frac{(q^{b_{ij}} z_2 / z_1; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_{\infty}}$  should be expanded in  $z_2 / z_1$ , whereas in the r.h.s  $\frac{(q^{b_{ij}} z_1 / z_2; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_{\infty}}$  should be expanded in  $z_1 / z_2$ . In each term in (2.20), the coefficient function is expanded in  $z_{\sigma(k)} / z_{\sigma(m)}$  ( $m < k$ ),  $w / z_{\sigma(m)}$  ( $m \leq s$ ) and  $z_{\sigma(m)} / w$  ( $m \geq s+1$ ). All the coefficients in  $z_j$ 's are well defined in the  $p$ -adic topology.

*Remark.* In [1, 2, 35, 36], assuming that  $q$  is a transcendental complex number satisfying  $|q| < 1$ , we wrote (2.17), (2.18) as

$$\begin{aligned}
z_1 \Theta_{p^*}(q^{b_{ij}} z_2 / z_1) e_i(z_1) e_j(z_2) &= -z_2 \Theta_{p^*}(q^{b_{ij}} z_1 / z_2) e_j(z_2) e_i(z_1), \\
z_1 \Theta_p(q^{-b_{ij}} z_2 / z_1) f_i(z_1) f_j(z_2) &= -z_2 \Theta_p(q^{-b_{ij}} z_1 / z_2) f_j(z_2) f_i(z_1),
\end{aligned}$$

in the sense of analytic continuation.

Let  $U_q(\widehat{\mathfrak{g}})$  be the quantum affine algebra associated with  $\widehat{\mathfrak{g}}$  in the Drinfeld realization [3]. See Appendix A.  $U_{q,p}(\widehat{\mathfrak{g}})$  is a natural face type (i.e. dynamical) elliptic deformation of  $U_q(\widehat{\mathfrak{g}})$  in the following sense.

**Theorem 2.2.**

$$U_{q,p}(\widehat{\mathfrak{g}}) / pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathbb{F} \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{g}})) \sharp \mathbb{C}[\mathcal{R}_Q].$$

Here the smash product  $\sharp$  is defined as follows.

$$\begin{aligned}
& g(P, P+h)x \otimes e^{\alpha} \cdot f(P, P+h)y \otimes e^{\beta} \\
& = g(P, P+h)f(P - \langle \alpha, P \rangle, P+h - \langle \alpha + \text{wt}(x), P+h \rangle)xy \otimes e^{\alpha+\beta}
\end{aligned}$$

where  $\text{wt}(x) \in \bar{\mathfrak{h}}^*$  s.t.  $q^h x q^{-h} = q^{\langle \text{wt}(x), h \rangle} x$  for  $x, y \in U_q(\widehat{\mathfrak{g}})$ ,  $f(P), g(P) \in \mathbb{F}$ ,  $e^{\alpha}, e^{\beta} \in \mathbb{C}[\mathcal{R}_Q]$ .

*Proof.* At  $p = 0$ , the relations for  $\alpha_{j,m}^{\vee}, e_j(z), f_j(z)$  (2.12)-(2.21) coincide with those for  $a_{i,m}^{\vee}, x_j^+(z), x_j^-(z)$  (A.6)-(A.11) of  $U_q(\widehat{\mathfrak{g}})$ . Therefore from (2.6)-(2.10), one has the isomorphism

$$e_j(z) \mapsto x_j^+(z) e^{-Q_{\alpha_j}}, \quad f_j(z) \mapsto x_j^-(z), \quad K_j^{\pm} \mapsto q_j^{\mp h_j} e^{-Q_{\alpha_j}}, \quad \alpha_{j,m}^{\vee} \mapsto a_{j,m}^{\vee} \pmod{pU_{q,p}(\widehat{\mathfrak{g}})}.$$

## 2.2 $H$ -algebra $U_{q,p}(\widehat{\mathfrak{g}})$

Let  $\mathcal{A}$  be a complex associative algebra,  $\mathcal{H}$  be a finite dimensional commutative subalgebra of  $\mathcal{A}$ , and  $\mathcal{M}_{\mathcal{H}^*}$  be the field of meromorphic functions on  $\mathcal{H}^*$  the dual space of  $\mathcal{H}$ .

**Definition 2.3** ( $\mathcal{H}$ -algebra). *An  $\mathcal{H}$ -algebra is an associative algebra  $\mathcal{A}$  with 1, which is bigraded over  $\mathcal{H}^*$ ,  $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha\beta}$ , and equipped with two algebra embeddings  $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{A}_{00}$  (the left and right moment maps), such that*

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathcal{A}_{\alpha\beta}, \quad \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where  $T_\alpha$  denotes the automorphism  $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$  of  $\mathcal{M}_{\mathcal{H}^*}$ .

**Proposition 2.4.**  $U = U_{q,p}(\widehat{\mathfrak{g}})$  is an  $H$ -algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta}$$

$$U_{\alpha, \beta} = \left\{ x \in U \mid q^{P+h} x q^{-(P+h)} = q^{<\alpha, P+h>} x, \quad q^P x q^{-P} = q^{<\beta, P>} x, \quad \forall P+h, P \in H \right\}$$

and  $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{0,0}$  defined by

$$\mu_l(\widehat{f}) = f(P+h, p) \in \mathbb{F}[[p]], \quad \mu_r(\widehat{f}) = f(P, p^*) \in \mathbb{F}[[p]].$$

## 2.3 Dynamical Representations

Let us consider a vector space  $\mathcal{V}$  over  $\mathbb{F}$ , which is  $H$ -diagonalizable, i.e.

$$\mathcal{V} = \bigoplus_{\lambda, \mu \in H^*} \mathcal{V}_{\lambda, \mu}, \quad \mathcal{V}_{\lambda, \mu} = \{v \in \mathcal{V} \mid q^{P+h} \cdot v = q^{<\lambda, P+h>} v, \quad q^P \cdot v = q^{<\mu, P>} v \quad \forall P+h, P \in H\}.$$

Let us define the  $H$ -algebra  $\mathcal{D}_{H, \mathcal{V}}$  of the  $\mathbb{C}$ -linear operators on  $\mathcal{V}$  by

$$\mathcal{D}_{H, \mathcal{V}} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \mathcal{V}})_{\alpha\beta},$$

$$(\mathcal{D}_{H, \mathcal{V}})_{\alpha\beta} = \left\{ X \in \text{End}_{\mathbb{C}} \mathcal{V} \mid \begin{array}{l} f(P+h)X = Xf(P+h+<\alpha, P+h>), \\ f(P)X = Xf(P+<\beta, P>), \\ f(P), f(P+h) \in \mathbb{F}, \quad X \cdot \mathcal{V}_{\lambda, \mu} \subseteq \mathcal{V}_{\lambda+\alpha, \mu+\beta} \end{array} \right\},$$

$$\mu_l^{\mathcal{D}_{H, \mathcal{V}}}(\widehat{f})v = f(<\lambda, P+h>, p)v, \quad \mu_r^{\mathcal{D}_{H, \mathcal{V}}}(\widehat{f})v = f(<\mu, P>, p^*)v, \quad \widehat{f} \in \mathcal{M}_{H^*}, \quad v \in \mathcal{V}_{\lambda, \mu}.$$

**Definition 2.5.** *We define a dynamical representation of  $U_{q,p}(\widehat{\mathfrak{g}})$  on  $\mathcal{V}$  to be an  $H$ -algebra homomorphism  $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H, \mathcal{V}}$ . By the action  $\pi$  of  $U_{q,p}(\widehat{\mathfrak{g}})$  we regard  $\mathcal{V}$  as a  $U_{q,p}(\widehat{\mathfrak{g}})$ -module.*

**Definition 2.6.** *For  $k \in \mathbb{C}$ , we say that a  $U_{q,p}(\widehat{\mathfrak{g}})$ -module has level  $k$  if  $c$  act as the scalar  $k$  on it.*

*Remark.* For the level-0 representations, Definition 2.5 is essentially the same as in [8], by identifying  $P$  and  $P + h$  with  $\lambda$  and  $\lambda - \gamma h$ , respectively. This definition is valid also for the non-zero level cases [10].

**Definition 2.7.** For  $\omega \in \mathbb{C}$ , we set

$$\mathcal{V}_\omega = \{v \in \mathcal{V} \mid d \cdot v = \omega v\}$$

and we call  $\mathcal{V}_\omega$  the space of elements homogeneous of degree  $\omega$ . We also say that  $X \in \mathcal{D}_{H,\mathcal{V}}$  is homogeneous of degree  $\omega \in \mathbb{C}$  if

$$[d, X] = \omega X$$

and denote by  $(\mathcal{D}_{H,\mathcal{V}})_\omega$  the space of all endomorphisms homogeneous of degree  $\omega$ .

**Definition 2.8.** Let  $\mathcal{H}, \mathcal{N}_+, \mathcal{N}_-$  be the subalgebras of  $U_{q,p}(\widehat{\mathfrak{g}})$  generated by  $c, d, K_i^\pm$  ( $i \in I$ ), by  $\alpha_{i,n}^\vee$  ( $i \in I, n \in \mathbb{Z}_{>0}$ ),  $e_{i,n}$  ( $i \in I, n \in \mathbb{Z}_{\geq 0}$ ),  $f_{i,n}$  ( $i \in I, n \in \mathbb{Z}_{>0}$ ) and by  $\alpha_{i,-n}^\vee$  ( $i \in I, n \in \mathbb{Z}_{>0}$ ),  $e_{i,-n}$  ( $i \in I, n \in \mathbb{Z}_{>0}$ ),  $f_{i,-n}$  ( $i \in I, n \in \mathbb{Z}_{\geq 0}$ ), respectively.

**Definition 2.9.** For  $k \in \mathbb{C}$ ,  $\lambda \in \mathfrak{h}^*$  and  $\mu \in H^*$ , a (dynamical)  $U_{q,p}(\widehat{\mathfrak{g}})$ -module  $\mathcal{V}(\lambda, \mu)$  is called the level- $k$  highest weight module with the highest weight  $(\lambda, \mu)$ , if there exists a vector  $v \in \mathcal{V}(\lambda, \mu)$  such that

$$\begin{aligned} \mathcal{V}(\lambda, \mu) &= U_{q,p}(\widehat{\mathfrak{g}}) \cdot v, & \mathcal{N}_+ \cdot v &= 0, \\ c \cdot v &= kv, & f(P) \cdot v &= f(< \mu, P >)v, & f(P + h) \cdot v &= f(< \lambda, P + h >)v. \end{aligned}$$

We define the category  $\mathfrak{C}_k$  in the analogous way to the classical affine Lie algebra case [18].

**Definition 2.10.** For  $k \in \mathbb{C}$ ,  $\mathfrak{C}_k$  is the full subcategory of the category of  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules consisting of those modules  $\mathcal{V}$  such that

- (i)  $\mathcal{V}$  has level  $k$
- (ii)  $\mathcal{V} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{V}_\omega$
- (iii) For every  $\omega \in \mathbb{C}$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$ ,  $\mathcal{V}_{\omega+n} = 0$ .

Since  $\pi \mathcal{N}_+ \subset \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{D}_{H,\mathcal{V}})_n$ , any level- $k$  highest weight  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules belong to  $\mathfrak{C}_k$ .

### 3 The Dynamical Quantum $Z$ -Algebras

In this section we introduce a quantum and dynamical analogue  $\mathcal{Z}_k$  of Lepowsky-Wilson's  $Z$ -algebra associated with the level- $k$   $U_{q,p}(\widehat{\mathfrak{g}})$ -modules and define a category  $\mathfrak{D}_k$  of the  $\mathcal{Z}_k$ -modules. Each representation of  $\mathcal{Z}_k$  in  $\mathfrak{D}_k$  turns out to be a dynamical analogue of the quantum  $Z$ -algebra derived by Jing [23] from the level- $k$  representation in the  $U_q(\widehat{\mathfrak{g}})$  counterpart of  $\mathfrak{D}_k$ . See sec.4.1. We also provide the Serre relations (3.23) which are not written in [23] explicitly.



### 3.1 The Heisenberg algebra $U_{q,p}(\mathcal{H})$

Let  $U_{q,p}(\mathcal{H})$  be the subalgebra of  $U_{q,p}(\widehat{\mathfrak{g}})$  generated by  $\alpha_{i,n}^\vee$  ( $i \in I, n \in \mathbb{Z}_{\neq 0}$ ) and  $c$ . It is convenient to introduce the simple root type generators  $\alpha_{j,m}$  and  $\alpha'_{j,m}$  defined by  $\alpha_{j,m} = [d_j]\alpha_{j,m}^\vee$  and  $\alpha'_{j,m} = \frac{1-p^{*m}}{1-p^m}q^{cm}\alpha_{j,m}$ , ( $j \in I, m \neq 0$ ). From (2.14), (2.15), (2.16), we have

$$[\alpha_{i,m}, \alpha_{j,n}] = \frac{[b_{ij}m][cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} \delta_{m+n,0}, \quad (3.1)$$

$$[\alpha'_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m][cm]}{m} \frac{1-p^{*m}}{1-p^m} q^{cm} \delta_{m+n,0}, \quad (3.2)$$

$$[\alpha_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m][cm]}{m} \delta_{m+n,0}, \quad (3.3)$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \quad (3.4)$$

$$[\alpha'_{i,m}, f_j(z)] = -\frac{[b_{ij}m]}{m} \frac{1-p^{*m}}{1-p^m} q^{cm} z^m f_j(z). \quad (3.5)$$

Let  $U_{q,p}(\mathcal{H}_+)$  (resp.  $U_{q,p}(\mathcal{H}_-)$ ) be the commutative subalgebras of  $U_{q,p}(\mathcal{H})$  generated by  $\{c, \alpha_{i,n} (i \in I, n \in \mathbb{Z}_{>0})\}$  (resp.  $\{\alpha_{i,-n} (i \in I, n \in \mathbb{Z}_{>0})\}$ ). We have

$$U_{q,p}(\mathcal{H}) = U_{q,p}(\mathcal{H}_-) U_{q,p}(\mathcal{H}_+).$$

Let  $\mathbb{C}1_k$  be the one-dimensional  $U_{q,p}(\mathcal{H}_+)$ -module generated by the vacuum vector  $1_k$  defined by

$$c \cdot 1_k = k1_k \quad \alpha_{i,n} \cdot 1_k = 0 \quad (n > 0).$$

Then we have the induced  $U_{q,p}(\mathcal{H})$ -module

$$\mathcal{F}_{\alpha,k} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathbb{C}1_k.$$

We identify  $\mathcal{F}_{\alpha,k}$  with a polynomial ring  $\mathbb{C}[\alpha_{i,-m} (i \in I, m > 0)]$  by

$$\begin{aligned} c \cdot u &= ku, \quad \alpha_{i,-n} \cdot u = \alpha_{i,-n} u, \\ \alpha_{i,n} \cdot u &= \sum_j \frac{[b_{ij}n][kn]}{n} \frac{1-p^n}{1-p^{*n}} q^{-kn} \frac{\partial}{\partial \alpha_{j,-n}} u \quad (n > 0) \end{aligned}$$

for  $u \in \mathbb{C}[\alpha_{i,-m} (i \in I, m > 0)]$ .

### 3.2 The dynamical quantum $Z$ -algebra $\mathcal{Z}_{\mathcal{V}}$

Let  $k \in \mathbb{C}^\times$  and  $(\mathcal{V}, \pi) \in \mathfrak{E}_k$ . We call  $\pi U_{q,p}(\mathcal{H}) \subset (D_{H,\mathcal{V}})_{00}$  the level- $k$  Heisenberg algebra. We define the following vertex operators in  $(D_{H,\mathcal{V}})_{00}[[z, z^{-1}]]$ .

$$E^\pm(\alpha_j, z) = \exp \left( \pm \sum_{n>0} \frac{\pi(\alpha_{j,\pm n})}{[kn]} z^{\mp n} \right), \quad E^\pm(\alpha'_j, z) = \exp \left( \mp \sum_{n>0} \frac{\pi(\alpha'_{j,\pm n})}{[kn]} z^{\mp n} \right).$$

These satisfy the following relations.

**Proposition 3.1.**

$$E^+(\alpha_i, z)E^-(\alpha_j, w) = \frac{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p^*)_\infty}{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p^*)_\infty} E^-(\alpha_j, w)E^+(\alpha_i, z), \quad (3.6)$$

$$E^+(\alpha'_i, z)E^-(\alpha'_j, w) = \frac{(q^{-b_{ij}}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p)_\infty}{(q^{b_{ij}}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p)_\infty} E^-(\alpha'_j, w)E^+(\alpha'_i, z), \quad (3.7)$$

$$E^+(\alpha_i, z)E^-(\alpha'_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha'_j, w)E^+(\alpha_i, z), \quad (3.8)$$

$$E^+(\alpha'_i, z)E^-(\alpha_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha_j, w)E^+(\alpha'_i, z), \quad (3.9)$$

$$E^\pm(\alpha_i, z)e_j(w) = \frac{(q^{\pm b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p^*)_\infty}{(q^{\mp b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p^*)_\infty} e_j(w)E^\pm(\alpha_i, z), \quad (3.10)$$

$$E^\pm(\alpha'_i, z)f_j(w) = \frac{(q^{\pm b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p)_\infty}{(q^{\mp b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p)_\infty} f_j(w)E^\pm(\alpha'_i, z), \quad (3.11)$$

$$E^\pm(\alpha'_i, z)e_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} e_j(w)E^\pm(\alpha'_i, z), \quad (3.12)$$

$$E^\pm(\alpha_i, z)f_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} f_j(w)E^\pm(\alpha_i, z). \quad (3.13)$$

**Definition 3.2.** We define  $\mathcal{Z}_j^\pm(z; \mathcal{V}) \in \mathcal{D}_{H, \mathcal{V}}[[z, z^{-1}]]$  by

$$\mathcal{Z}_j^+(z; \mathcal{V}) := E^-(\alpha_j, z)\pi(e_j(z))E^+(\alpha_j, z), \quad (3.14)$$

$$\mathcal{Z}_j^-(z; \mathcal{V}) := E^-(\alpha'_j, z)\pi(f_j(z))E^+(\alpha'_j, z). \quad (3.15)$$

for  $j \in I$  and call them the dynamical quantum  $Z$  operators associated with  $(\mathcal{V}, \pi) \in \mathfrak{E}_k$ .

Note that due to the truncation property of the grading of  $\mathcal{V} \in \mathfrak{E}_k$  w.r.t  $d$ ,  $\mathcal{Z}_j^\pm(z; \mathcal{V})$  are well defined i.e. the coefficients  $\mathcal{Z}_{j,n}^\pm(\mathcal{V})$  of  $\mathcal{Z}_j^\pm(z; \mathcal{V}) = \sum_{n \in \mathbb{Z}} \mathcal{Z}_{j,n}^\pm(\mathcal{V})z^{-n}$  in  $z$  are well defined elements in  $(\mathcal{D}_{H, \mathcal{V}})_n$  for all  $n \in \mathbb{Z}$ . For the sake of simplicity of the presentation, we often drop  $\pi$  to denote the elements in  $\mathcal{D}_{H, \mathcal{V}}$ .

From the defining relations of  $U_{q,p}(\widehat{\mathfrak{g}})$ , we obtain the following relations of the dynamical quantum  $Z$  operators.

**Theorem 3.3.**

$$g(P+h)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P+h), \quad g(P)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P - \langle Q_{\alpha_i}, P \rangle), \quad (3.16)$$

$$g(P+h)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P+h - \langle \alpha_i, P+h \rangle), \quad g(P)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P), \quad (3.17)$$

$$[d, \mathcal{Z}_j^\pm(z; \mathcal{V})] = -z \frac{\partial}{\partial z} \mathcal{Z}_j^\pm(z; \mathcal{V}), \quad (3.18)$$

$$[\alpha_{i,m}, \mathcal{Z}_j^\pm(w; \mathcal{V})] = 0, \quad (3.19)$$

$$K_i^\pm \mathcal{Z}_j^+(z; \mathcal{V}) = q^{\mp b_{ij}} \mathcal{Z}_j^+(z; \mathcal{V}) K_i^\pm, \quad K_i^\pm \mathcal{Z}_j^-(z; \mathcal{V}) = q^{\pm b_{ij}} \mathcal{Z}_j^-(z; \mathcal{V}) K_i^\pm, \quad (3.20)$$

$$z \frac{(q^{-b_{ij}}w/z; q^{2k})_\infty}{(q^{b_{ij}+2k}w/z; q^{2k})_\infty} \mathcal{Z}_i^\pm(z; \mathcal{V}) \mathcal{Z}_j^\pm(w; \mathcal{V}) = -w \frac{(q^{-b_{ij}}z/w; q^{2k})_\infty}{(q^{b_{ij}+2k}z/w; q^{2k})_\infty} \mathcal{Z}_j^\pm(w; \mathcal{V}) \mathcal{Z}_i^\pm(z; \mathcal{V}), \quad (3.21)$$

$$\begin{aligned} & \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^-(w; \mathcal{V}) - \frac{(q^{b_{ij}+k}z/w; q^{2k})_\infty}{(q^{-b_{ij}+k}z/w; q^{2k})_\infty} \mathcal{Z}_j^-(w; \mathcal{V}) \mathcal{Z}_i^+(z; \mathcal{V}) \\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left( K_i^- \delta(q^{-k}z/w) - K_i^+ \delta(q^kz/w) \right), \end{aligned} \quad (3.22)$$

$$\begin{aligned} & \sum_{\sigma \in S_a} \prod_{1 \leq m < l \leq a} \frac{(q^{2+k \mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty}{(q^{-2+k \mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty} \\ & \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(q^{-b_{ij}+k \mp k} w/z_{\sigma(m)}; q^{2k})_\infty}{(q^{b_{ij}+k \mp k} w/z_{\sigma(m)}; q^{2k})_\infty} \prod_{s+1 \leq m \leq a} \frac{(q^{-b_{ij}+k \mp k} z_{\sigma(m)}/w; q^{2k})_\infty}{(q^{b_{ij}+k \mp k} z_{\sigma(m)}/w; q^{2k})_\infty} \\ & \times \mathcal{Z}_i^\pm(z_{\sigma(1)}; \mathcal{V}) \cdots \mathcal{Z}_i^\pm(z_{\sigma(s)}; \mathcal{V}) \mathcal{Z}_j^\pm(w; \mathcal{V}) \mathcal{Z}_i^\pm(z_{\sigma(s+1)}; \mathcal{V}) \cdots \mathcal{Z}_i^\pm(z_{\sigma(a)}; \mathcal{V}) = 0 \\ & (i \neq j, a = 1 - a_{ij}). \end{aligned} \quad (3.23)$$

*Proof.* The relations (3.17) and (3.18) follow from (2.6)-(2.10) and (2.12), respectively. Let us show the relation (3.19). For  $m > 0$ , we have

$$[\alpha_{i,m}, \mathcal{Z}_j^+(z; \mathcal{V})] = [\alpha_{i,m}, E^-(\alpha_j, z)] e_j(z) E^+(\alpha_j, z) + E^-(\alpha_j, z) [\alpha_{i,m}, e_j(z)] E^+(\alpha_j, z).$$

This vanishes due to (3.4) and

$$[\alpha_{i,m}, E^-(\alpha_j, z)] = -\frac{[b_{ij}m]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-km} z^m,$$

where  $p^* = pq^{-2k}$ . Similarly,  $[\alpha_{i,m}, \mathcal{Z}_j^-(z; \mathcal{V})] = 0$  follows from (3.5) and

$$[\alpha'_{i,m}, E^-(\alpha'_j, z)] = \frac{[b_{ij}m]}{m} \frac{1 - p^{*m}}{1 - p^m} q^{km} z^m.$$

The case  $m < 0$  can be proved in a similar way.

The relation (3.21) follows from

$$\begin{aligned} & \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^+(w; \mathcal{V}) \\ &= E^-(\alpha_i, z) e_i(z) E^+(\alpha_i, z) E^-(\alpha_j, w) e_j(w) E^+(\alpha_j, w) \\ &= \frac{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p^*)_\infty}{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p^*)_\infty} E^-(\alpha_i, z) e_i(z) E^-(\alpha_j, w) E^+(\alpha_i, z) e_j(w) E^+(\alpha_j, w) \\ &= \frac{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p^*)_\infty}{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p^*)_\infty} E^-(\alpha_i, z) E^-(\alpha_j, w) e_i(z) e_j(w) E^+(\alpha_i, z) E^+(\alpha_j, w) \\ &= -\frac{w}{z(1 - q^{-b_{ij}}w/z)} \frac{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}z/w; p^*)_\infty}{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (p^*q^{-b_{ij}}z/w; p^*)_\infty} \\ & \quad \times E^-(\alpha_i, z) E^-(\alpha_j, w) e_j(w) e_i(z) E^+(\alpha_i, z) E^+(\alpha_j, w) \\ &= -\frac{w(1 - q^{-b_{ij}}z/w)}{z(1 - q^{-b_{ij}}w/z)} \frac{(q^{-b_{ij}+2k}z/w; q^{2k})_\infty^2 (q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}z/w; p^*)_\infty}{(q^{b_{ij}+2k}z/w; q^{2k})_\infty^2 (q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}z/w; p^*)_\infty} \\ & \quad \times E^-(\alpha_j, w) e_j(w) E^-(\alpha_i, z) E^+(\alpha_j, w) e_i(z) E^+(\alpha_i, z) \\ &= -\frac{w}{z} \frac{(q^{-b_{ij}}z/w; q^{2k})_\infty (q^{b_{ij}+2k}w/z; q^{2k})_\infty}{(q^{b_{ij}+2k}z/w; q^{2k})_\infty (q^{-b_{ij}}w/z; q^{2k})_\infty} \mathcal{Z}_j^+(w; \mathcal{V}) \mathcal{Z}_i^+(z; \mathcal{V}). \end{aligned}$$

We also derive (3.22) as follows.

$$\begin{aligned}
& \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^-(w; \mathcal{V}) \\
&= \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha_i, z) e_i(z) E^+(\alpha_i, z) E^-(\alpha'_j, w) f_j(w) E^+(\alpha'_j, w) \\
&= E^-(\alpha_i, z) E^-(\alpha'_j, w) e_i(z) f_j(w) E^+(\alpha_i, z) E^+(\alpha'_j, w) \\
&= E^-(\alpha_i, z) E^-(\alpha'_j, w) \left[ f_j(w) e_i(z) + \frac{\delta_{ij}}{q_i - q_i^{-1}} \left( \delta \left( q^{-k} \frac{z}{w} \right) \psi_i^-(q^{k/2} w) - \delta \left( q^k \frac{z}{w} \right) \psi_i^+(q^{-k/2} w) \right) \right] \\
&\quad \times E^+(\alpha_i, z) E^+(\alpha'_j, w).
\end{aligned}$$

Then use

$$\psi_i^\pm(q^{\mp k/2} w) = K_i^\pm E^-(\alpha_i, q^{\mp k} w)^{-1} E^-(\alpha'_i, w)^{-1} E^+(\alpha_i, q^{\mp k} w)^{-1} E^+(\alpha'_i, w)^{-1}$$

and the property of the delta function.

To prove the Serre relation (3.23) for  $\mathcal{Z}_j^+(z)$  we use (3.14) and (3.19) and obtain

$$e_i(z) = E(\alpha_i, z) \mathcal{Z}_i^+(z; \mathcal{V}) \quad (3.24)$$

where we set

$$E(\alpha_i, z) = E^-(\alpha_i, z)^{-1} E^+(\alpha_i, z)^{-1}.$$

From (3.6), we have

$$\begin{aligned}
& E(\alpha_i, z) E(\alpha_j, w) \\
&= \frac{(q^{-2}w/z; q^{2k})_\infty (q^2z/w; q^{2k})_\infty (p^*q^{-2}w/z; p^*)_\infty (p^*q^2z/w; p^*)_\infty}{(q^2w/z; q^{2k})_\infty (q^{-2}z/w; q^{2k})_\infty (p^*q^2w/z; p^*)_\infty (p^*q^{-2}z/w; p^*)_\infty} E(\alpha_j, w) E(\alpha_i, z).
\end{aligned} \quad (3.25)$$

Next note that (2.20) is equivalent to

$$\begin{aligned}
0 &= \prod_{1 \leq m < l \leq a} \frac{(p^*q^2z_l/z_m; p^*)_\infty}{(p^*q^{-2}z_l/z_m; p^*)_\infty} \prod_{1 \leq i \leq a} \frac{(p^*q^{b_{ij}}z_i/w; p^*)_\infty}{(p^*q^{-b_{ij}}z_i/w; p^*)_\infty} \\
&\quad \times \sum_{\sigma \in S_a} \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(p^*q^{-2}z_l/z_m; p^*)_\infty}{(p^*q^2z_l/z_m; p^*)_\infty} \frac{(p^*q^2z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-2}z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty} \\
&\quad \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^*q^{b_{ij}}w/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-b_{ij}}w/z_{\sigma(m)}; p^*)_\infty} \frac{(p^*q^{-b_{ij}}z_{\sigma(m)}/w; p^*)_\infty}{(p^*q^{b_{ij}}z_{\sigma(m)}/w; p^*)_\infty} \\
&\quad \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}).
\end{aligned}$$

Substitute (3.24) into this, and move all  $E(\alpha_i, z_j)$  and  $E(\alpha_j, w)$  to the left. Then we get

$$\begin{aligned}
0 &= \prod_{1 \leq m < l \leq a} \frac{(p^* q^2 z_l / z_m; p^*)_\infty}{(p^* q^{-2} z_l / z_m; p^*)_\infty} \prod_{1 \leq i \leq a} \frac{(p^* q^{b_{ij}} z_i / w; p^*)_\infty}{(p^* q^{-b_{ij}} z_i / w; p^*)_\infty} \\
&\times \sum_{\sigma \in S_a} \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(p^* q^{-2} z_l / z_m; p^*)_\infty}{(p^* q^2 z_l / z_m; p^*)_\infty} \frac{(p^* q^2 z_{\sigma(l)} / z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(l)} / z_{\sigma(m)}; p^*)_\infty} \\
&\times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_\infty} \frac{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_\infty}{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_\infty} \\
&\times \varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(a)}) \\
&\times \mathcal{Z}_i^+(z_{\sigma(1)}; \mathcal{V}) \cdots \mathcal{Z}_i^+(z_{\sigma(s)}; \mathcal{V}) \mathcal{Z}_j^+(w; \mathcal{V}) \mathcal{Z}_i^+(z_{\sigma(s+1)}; \mathcal{V}) \cdots \mathcal{Z}_i^+(z_{\sigma(a)}; \mathcal{V}), \quad (3.26)
\end{aligned}$$

where we set

$$\begin{aligned}
&\varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(a)}) \\
&= E(\alpha_i, z_{\sigma(1)}) \cdots E(\alpha_i, z_{\sigma(s)}) E(\alpha_j, w) E(\alpha_i, z_{\sigma(s+1)}) \cdots E(\alpha_i, z_{\sigma(a)}).
\end{aligned}$$

Then moving  $E(\alpha_j, w)$  to the left end by (3.25), we have

$$\begin{aligned}
&\varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(N)}) \\
&= \prod_{1 \leq i \leq s} \frac{(q^{-b_{ij}} w / z_{\sigma(i)}; q^{2k})_\infty (q^{b_{ij}} z_{\sigma(i)} / w; q^{2k})_\infty (p^* q^{-b_{ij}} w / z_{\sigma(i)}; p^*)_\infty (p^* q^{b_{ij}} z_{\sigma(i)} / w; p^*)_\infty}{(q^{b_{ij}} w / z_{\sigma(i)}; q^{2k})_\infty (q^{-b_{ij}} z_{\sigma(i)} / w; q^{2k})_\infty (p^* q^{b_{ij}} w / z_{\sigma(i)}; p^*)_\infty (p^* q^{-b_{ij}} z_{\sigma(i)} / w; p^*)_\infty} \\
&\times \varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)}).
\end{aligned}$$

Substituting this into (3.26), we can factor out  $\varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)})$  from  $\sum_{s=0}^a$ . Then exchanging the order of  $E(\alpha_i, z_l)$ 's by (3.25), we have

$$\begin{aligned}
&\varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)}) \\
&= \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(q^{-2} z_l / z_m; q^{2k})_\infty (q^2 z_{\sigma(l)} / z_{\sigma(m)}; q^{2k})_\infty (p^* q^2 z_l / z_m; p^*)_\infty (q^{-2} p^* z_{\sigma(l)} / z_{\sigma(m)}; p^*)_\infty}{(q^2 z_l / z_m; q^{2k})_\infty (q^{-2} z_{\sigma(l)} / z_{\sigma(m)}; q^{2k})_\infty (p^* q^{-2} z_l / z_m; p^*)_\infty (p^* q^2 z_{\sigma(l)} / z_{\sigma(m)}; p^*)_\infty} \\
&\times \varepsilon(w, z_1, \dots, z_a).
\end{aligned}$$

Substituting this into (3.26), we can factor out  $\varepsilon(w, z_1, \dots, z_a)$  completely from  $\sum_{\sigma \in S_a}$ . Multiply

$$\prod_{1 \leq m < l \leq a} \frac{(q^2 z_l / z_m; q^{2k})_\infty}{(q^{-2} z_l / z_m; q^{2k})_\infty} \prod_{1 \leq m \leq a} \frac{(q^{-b_{ij}} z_m / w; q^{2k})_\infty}{(q^{b_{ij}} z_m / w; q^{2k})_\infty},$$

and drop the overall factor depending on  $p^*$ , one gets the desired relation. One can prove the  $\mathcal{Z}_j^-(z; \mathcal{V})$  case in the same way.

**Definition 3.4.** For  $k \in \mathbb{C}^\times$  and  $(\mathcal{V}, \pi) \in \mathfrak{C}_k$ , we call the  $H$ -subalgebra of  $\mathcal{D}_{H, \mathcal{V}}$  generated by  $\mathcal{Z}_{i, m}^\pm(\mathcal{V})$ ,  $K_i^\pm$  ( $i \in I, m \in \mathbb{Z}$ ),  $\mathcal{M}_{H^*}$  and  $d$  the dynamical quantum  $Z$ -algebra  $\mathcal{Z}_{\mathcal{V}}$  associated with  $(\mathcal{V}, \pi)$ .

### 3.3 The universal algebra $\mathcal{Z}_k$

Using the relations in Theorem 3.3, we define the universal dynamical quantum  $Z$ -algebra as follows.

**Definition 3.5.** Let  $\mathcal{Z}_{i,m}^\pm$  ( $i \in I, m \in \mathbb{Z}$ ) be abstract symbols. We set  $\mathcal{Z}_i^\pm(z) = \sum_{m \in \mathbb{Z}} \mathcal{Z}_{i,m}^\pm z^{-m}$ . We define the universal dynamical quantum  $Z$ -algebra  $\mathcal{Z}_k$  to be a topological algebra over  $\mathbb{F}[[q^{2k}]]$  generated by  $\mathcal{Z}_{i,m}^\pm, K_i^\pm$  ( $i \in I, m \in \mathbb{Z}$ ),  $d, \mathcal{M}_{H^*}$  subject to the relations obtained by replacing  $\mathcal{Z}_i^\pm(z; \mathcal{V})$  by  $\mathcal{Z}_i^\pm(z)$  in Theorem 3.3.

We treat the relations as formal Laurent series in  $z, w$  and  $z_j$ 's in a similar way to those of  $U_{q,p}(\widehat{\mathfrak{g}})$  in Sec.2.1. The defining relations are well-defined in the  $q^{2k}$ -adic topology.

**Proposition 3.6.**  $\mathcal{Z}_k$  is an  $H$ -algebra with the same  $\mu_l, \mu_r$  as in  $U_{q,p}(\widehat{\mathfrak{g}})$ .

Note that for  $(\mathcal{V}, \pi) \in \mathfrak{C}_k$  we extend  $\pi$  to the map  $\pi : \mathcal{Z}_k \rightarrow \mathcal{D}_{H,\mathcal{V}}$  by  $\pi(\mathcal{Z}_{i,m}^\pm) = \mathcal{Z}_{i,m}^\pm(\mathcal{V})$ . Then  $\mathcal{V}$  is a  $\mathcal{Z}_k$ -module by  $\pi$ .

**Definition 3.7.** For  $k \in \mathbb{C}^\times$ , we denote by  $\mathfrak{D}_k$  the full subcategory of the category of  $\mathcal{Z}_k$ -modules consisting of those modules  $(\mathcal{W}, \sigma)$  such that

- (i)  $\mathcal{W}$  has level  $k$ .
- (ii)  $\mathcal{W} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{W}_\omega$ , where  $\mathcal{W}_\omega = \{w \in \mathcal{W} \mid \sigma(d)w = \omega w\}$
- (iii) For every  $\omega \in \mathbb{C}$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$ ,  $\mathcal{W}_{\omega+n} = 0$ .

Let us consider  $(\mathcal{V}, \pi) \in \mathfrak{C}_k$ . Following Lepowsky and Wilson [18], we define the vacuum space  $\Omega_{\mathcal{V}}$  by

$$\Omega_{\mathcal{V}} = \{v \in \mathcal{V} \mid \pi(\alpha_{i,n})v = 0 \quad \forall i \in I, n \in \mathbb{Z}_{>0}\}.$$

From Theorem 3.3,  $\Omega_{\mathcal{V}}$  is stable under the action of  $\mathcal{Z}_{\mathcal{V}}$ . For a morphism  $f : \mathcal{V} \rightarrow \mathcal{V}'$  in  $\mathfrak{C}_k$ , we have

$$f(\Omega_{\mathcal{V}}) \subset \Omega_{\mathcal{V}'}.$$

**Proposition 3.8.** For  $(\mathcal{V}, \pi) \in \mathfrak{C}_k$ , there is a unique representation  $\sigma$  of  $\mathcal{Z}_k$  on  $\Omega_{\mathcal{V}}$  such that  $(\Omega_{\mathcal{V}}, \sigma) \in \mathfrak{D}_k$ ,

$$\sigma(K_i^\pm) = \pi(K_i^\pm), \quad \sigma(\mathcal{Z}_{i,m}^\pm) = \mathcal{Z}_{i,m}^\pm(\mathcal{V}) \quad \forall i \in I, m \in \mathbb{Z}.$$

We hence define a functor  $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$  by

$$\Omega(\mathcal{V}, \pi) = (\Omega_{\mathcal{V}}, \sigma), \quad \Omega(f) = f|_{\Omega_{\mathcal{V}}} : \Omega_{\mathcal{V}} \rightarrow \Omega_{\mathcal{V}'}.$$

### 3.4 The functor $\Lambda$

We define a reverse functor  $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$  as follows. Let  $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$  be a  $\mathcal{Z}_k$ -module. We define  $U_{q,p}(\mathcal{H})$ -module  $\text{Ind } \mathcal{W}$  by requiring  $\alpha_{i,m} \cdot \mathcal{W} = 0$  and

$$\text{Ind } \mathcal{W} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathcal{W}.$$

Let  $\mathcal{F}_{\alpha,k}$  be the level- $k$  Fock module defined in Sec.3.1. We have a natural isomorphism  $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \cong \text{Ind } \mathcal{W}$  by  $(u \otimes 1_k) \otimes w \mapsto u \otimes w$  [18]. We thus identify the  $U_{q,p}(\mathcal{H})$ -module  $\text{Ind } \mathcal{W}$  with  $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$ , with the action  $\pi$  of  $U_{q,p}(\mathcal{H})$

$$\pi(c) = 1 \otimes c, \quad \pi(K_i^{\pm}) = 1 \otimes \sigma(K_i^{\pm}), \quad \pi(\alpha_{i,m}) = \alpha_{i,m} \otimes 1.$$

For  $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$  and  $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$ , we define  $e'_j(z), f'_j(z) \in \mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$  by

$$\begin{aligned} e'_j(z) &= E^-(\alpha_j, z)^{-1} E^+(\alpha_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^+(z)), \\ f'_j(z) &= E^-(\alpha'_j, z)^{-1} E^+(\alpha'_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^-(z)). \end{aligned}$$

These are well-defined elements of  $\mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$ . By a similar argument to the proof of Theorem 3.3 one can show that  $e'_j(z)$  and  $f'_j(z)$  satisfy the defining relations of  $U_{q,p}(\widehat{\mathfrak{g}})$  with  $c = k$ . We hence extend  $\pi : U_{q,p}(\mathcal{H}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$  to  $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$  as an  $H$ -algebra homomorphism by

$$\begin{aligned} \pi(e_j(z)) &= e'_j(z), \quad \pi(f_j(z)) = f'_j(z), \\ \pi(d) &= d \otimes 1 + 1 \otimes \sigma(d). \end{aligned}$$

By construction, the latter map is uniquely determined.

**Proposition 3.9.** *For  $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$ , there is a unique level- $k$   $U_{q,p}(\widehat{\mathfrak{g}})$ -module  $(\text{Ind } \mathcal{W}, \pi) \in \mathfrak{C}_k$ .*

We thus reach the following definition.

**Definition 3.10.** *We define a functor  $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$  by*

$$(i) \quad \Lambda(\mathcal{W}, \sigma) = (\text{Ind } \mathcal{W}, \pi)$$

(ii) *For a morphism  $f : \mathcal{W} \rightarrow \mathcal{W}'$  in  $\mathfrak{D}_k$ , define  $\Lambda(f) : \text{Ind } \mathcal{W} \rightarrow \text{Ind } \mathcal{W}'$  to be the induced  $U_{q,p}(\mathcal{H})$ -module map. Then  $\Lambda(f)$  is a  $U_{q,p}(\widehat{\mathfrak{g}})$ -module map.*

We obtain the following theorem analogously to the case of the affine Lie algebras [18].

**Theorem 3.11.** *For  $k \in \mathbb{C}^\times$ , the two categories  $\mathfrak{C}_k$  and  $\mathfrak{D}_k$  are equivalent by the functors  $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$  and  $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$ . In particular, the level- $k$   $U_{q,p}(\widehat{\mathfrak{g}})$ -module  $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \in \mathfrak{C}_k$  is irreducible if and only if  $\mathcal{W} \in \mathfrak{D}_k$  is an irreducible  $\mathcal{Z}_k$ -module.*

## 4 The Induced $U_{q,p}(\widehat{\mathfrak{g}})$ -Modules

In this section we give a simple realization of the dynamical quantum  $Z$ -algebra  $\mathcal{Z}_k$  in terms of the quantum  $Z$ -algebra  $Z_k$  associated with  $U_q(\widehat{\mathfrak{g}})$  and construct the level- $k$  induced  $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. We also give some examples of the level-1 irreducible representations.

### 4.1 The quantum $Z$ -algebra $Z_k$ associated with $U_q(\widehat{\mathfrak{g}})$

One can apply the arguments similar to those in Secs.3.1-3.3 to the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  in the Drinfeld realization and define the corresponding quantum  $Z$ -algebras  $Z_V$  associated with the level- $k$   $U_q(\widehat{\mathfrak{g}})$ -module  $V$  [23] and the universal one  $Z_k$ . See Appendix A. We also denote by  $C_k$  and  $D_k$  the  $U_q(\widehat{\mathfrak{g}})$  counterparts of the categories  $\mathfrak{C}_k$  and  $\mathfrak{D}_k$ .

Comparing the defining relations of  $\mathcal{Z}_k$  with those of  $Z_k$ , we obtain the following isomorphism.

**Proposition 4.1.** *We have the isomorphism*

$$\mathcal{Z}_k \cong (\mathbb{F} \otimes_{\mathbb{C}} Z_k) \sharp \mathbb{C}[\mathcal{R}_Q]$$

as an  $H$ -algebra by

$$\begin{aligned} \mathcal{Z}_{j,m}^+ &\mapsto Z_{j,m}^+ e^{-Q_{\alpha_j}}, & \mathcal{Z}_{j,m}^- &\mapsto Z_{j,m}^-, \\ K_i^{\pm} &\mapsto q_i^{\mp h_i} e^{-Q_{\alpha_j}} \quad (i \in I, m \in \mathbb{Z}), & d &\mapsto \bar{d}, \end{aligned}$$

where  $Z_{j,m}^{\pm}$  denotes the generators in  $Z_k$  (Definition A.3).

**Theorem 4.2.** *For  $(W, \bar{\sigma}) \in D_k$  and generic  $\mu \in \mathfrak{h}^*$ , there is a dynamical representation  $\sigma$  of  $\mathcal{Z}_k$  on  $\mathcal{W}_{H,Q}(\mu) := (\mathbb{F} \otimes_{\mathbb{C}} W) \otimes_{\mathbb{C}} e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$  such that  $(\mathcal{W}_{H,Q}(\mu), \sigma) \in \mathfrak{C}_k$  and*

$$\begin{aligned} \sigma(\mathcal{Z}_{j,m}^+) &= \bar{\sigma}(Z_{j,m}^+) \otimes e^{-Q_{\alpha_j}}, & \sigma(\mathcal{Z}_{j,m}^-) &= \bar{\sigma}(Z_{j,m}^-) \otimes 1, \\ \sigma(K_j^{\pm}) &= \bar{\sigma}(q_j^{\mp h_j}) \otimes e^{-Q_{\alpha_j}}, & \sigma(d) &= \bar{\sigma}(\bar{d}) \otimes 1 + 1 \otimes P_d, \end{aligned}$$

where  $P_d$  denotes a  $\mathbb{C}$ -linear operator on  $1 \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$  such that

$$[1 \otimes P_d, \sigma(\mathcal{Z}_{j,m}^{\pm})] = 0.$$

**Proposition 4.3.** *The representation  $(\mathcal{W}_{H,Q}(\mu), \sigma)$  of  $\mathcal{Z}_k$  is irreducible if and only if  $W$  is an irreducible  $Z_k$ -module.*

From this and Theorem 3.11, we obtain:

**Proposition 4.4.** *For a  $Z_k$ -module  $(W, \bar{\sigma}) \in D_k$  and generic  $\mu \in \mathfrak{h}^*$ , let  $(\mathcal{W}_{H,Q}(\mu), \sigma)$  be the  $\mathcal{Z}_k$ -module constructed in Theorem 4.2 and  $\text{Ind } \mathcal{W}_{H,Q}(\mu) = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}_{H,Q}(\mu)$  be the level- $k$  induced  $U_{q,p}(\widehat{\mathfrak{g}})$ -module given in Proposition 3.9. Then  $(\text{Ind } \mathcal{W}_{H,Q}(\mu), \pi)$  is irreducible if and only if  $(W, \bar{\sigma})$  is irreducible.*



## 4.2 Examples of the irreducible representations

We here give some examples of the level-1 irreducible induced representations of  $U_{q,p}(\widehat{\mathfrak{g}})$  of types  $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$  and  $B_l^{(1)}$ .

### 4.2.1 The simply laced case :

Let  $\mathbb{C}[\mathcal{Q}]$  be the group algebra of the root lattice  $\mathcal{Q} = \oplus_i \mathbb{Z}\alpha_i$  with the central extension:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad (i, j \in I).$$

Let us consider the fundamental weight  $\Lambda_a$  of  $\widehat{\mathfrak{g}}$  with  $0 \leq a \leq l$  for  $A_l^{(1)}$ ,  $a = 0, 1, l-1, l$  for  $D_l^{(1)}$ ,  $a = 0, 1, 2$  for  $E_6^{(1)}$ ,  $a = 0, 1$  for  $E_7^{(1)}$ ,  $a = 0$  for  $E_8^{(1)}$ .

**Theorem 4.5.** [23, 39] *An inequivalent set of the level-1 irreducible  $Z_1(\widehat{\mathfrak{g}})$ -modules is given by  $W(\Lambda_a) = e^{\bar{\Lambda}_a} \mathbb{C}[\mathcal{Q}]$ , on which the actions of  $Z_j^\pm(z)$  are given by*

$$Z_j^\pm(z) = e^{\pm \alpha_j} z^{\pm h_j + 1} \quad (4.1)$$

with

$$z^{\pm h_i} e^{\pm \alpha_j} e^{\bar{\Lambda}_a} = z^{\pm(\alpha_i^\vee, \alpha_j + \bar{\Lambda}_a)} e^{\pm \alpha_j} e^{\bar{\Lambda}_a} \quad (i, j \in I).$$

Then for generic  $\mu \in \mathfrak{h}^*$ , we have from Theorem 4.2 a level-1 irreducible  $Z_1(\widehat{\mathfrak{g}})$  module  $\mathcal{W}_{H,Q}(\Lambda_a, \mu) := (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$  with the action given by

$$\mathcal{Z}_j^+(z) = Z_j^+(z) \otimes e^{-Q_{\alpha_j}}, \quad \mathcal{Z}_j^-(z) = Z_j^-(z) \otimes 1. \quad (4.2)$$

Then from Proposition 4.4 we obtain:

**Theorem 4.6.** *A level-1 irreducible highest weight representations of  $U_{q,p}(\widehat{\mathfrak{g}})$  is given by  $\mathcal{V}(\Lambda_a + \mu, \mu) := \text{Ind } \mathcal{W}_{H,Q}(\Lambda_a, \mu)$  with the highest weight  $(\Lambda_a + \mu, \mu)$ :*

$$\mathcal{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha,1} \otimes \mathcal{W}_{H,Q}(\Lambda_a, \mu) = \bigoplus_{\gamma, \kappa \in \mathcal{Q}} \mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu),$$

where

$$\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu) = \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes e^{\bar{\Lambda}_a + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}},$$

The highest weight vector is  $1_1 \otimes e^{\bar{\Lambda}_a} \otimes e^{Q_{\bar{\mu}}}$ . The derivation operator  $d$  is realized as

$$\begin{aligned} d &= -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2) (P + h)^j, \\ N^\alpha &= \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j, \end{aligned}$$

where  $r, r^* \in \mathbb{C}^\times$ , and  $A_m^j$  are the fundamental weight type elliptic bosons given in Sec.5.1.

One can easily calculate the character of  $\mathcal{V}(\Lambda_a, \mu)$ :

$$\begin{aligned} ch_{\mathcal{V}(\Lambda_a+\mu, \mu)} &= \text{tr}_{\mathcal{V}(\Lambda_a+\mu, \mu)} q^{-d-\frac{c(W(\mathfrak{g}))}{24}} = \sum_{\gamma, \kappa \in \mathcal{Q}} ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)}, \\ ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)} &= \frac{1}{\eta(q)^l} q^{\frac{1}{2rr^*}|r(\bar{\mu}+\kappa+\rho)-r^*(\bar{\Lambda}_a+\bar{\mu}+\gamma+\kappa+\rho)|^2}. \end{aligned}$$

Here  $c(W(\mathfrak{g})) = l(1 - \frac{g(g+1)}{rr^*})$ , and  $\eta(q)$  denotes Dedekind's  $\eta$ -function given by

$$\eta(q) = q^{\frac{1}{24}}(q; q)_{\infty}.$$

One should note that the character  $ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)}$  coincides with the one of the Verma module of the  $W(\mathfrak{g})$ -algebras for  $\mathfrak{g} = A_l, D_l, E_6, E_7, E_8$  with the highest weight  $h = \frac{1}{2rr^*}|r(\bar{\mu} + \kappa + \rho) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \rho)|^2$  and the central charge  $c(W(\mathfrak{g}))$ . In fact, for  $\widehat{\mathfrak{g}} = A_l^{(1)}$  case, for example, one can construct an action of the deformed  $W(A_l)$  algebra on  $\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)$  explicitly.

**Theorem 4.7.** [26, 27] For  $p = q^{2r}$  and  $p^* = pq^{-2} = q^{2r^*}$ , i.e.  $r^* = r - 1$ , the deformed  $W(A_l)$ -algebra acts on  $\mathcal{F}_{\xi, \kappa}(\Lambda_i + \mu, \mu)$  by

$$\begin{aligned} \Lambda_j(z) &=: \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m})(1 - p^{*m}) \mathcal{E}_m^{+j} (q^j z)^{-m} \right\} : \otimes p^{*h_{\bar{e}_j}} \quad (1 \leq j \leq l), \\ T_n(z) &= \sum_{1 \leq j_1 < \dots < j_n \leq l} : \Lambda_{j_1}(z) \Lambda_{j_2}(zq^{-2}) \dots \Lambda_{j_n}(zq^{-2(n-1)}) : \quad (1 \leq n \leq l). \end{aligned}$$

Here  $\mathcal{E}_m^{\pm j}$  denotes the orthonormal basis type elliptic boson given in (5.3), and  $:$  denotes the normal ordering of the enclosed expression such that the operators  $\mathcal{E}_m^{\pm j}$  for  $m < 0$  are to be placed to the left of the operators  $\mathcal{E}_m^{\pm j}$  for  $m > 0$ . In addition, the level-1 elliptic currents  $e_j(w)$  and  $f_j(w)$  of  $U_{q,p}(A_l^{(1)})$  obtained from Proposition 3.9, (4.1) and (4.2) are the screening currents of the deformed  $W(A_l)$ -algebra, i.e. they commute with  $T_n(z)$  up to a total difference.

See also [2, 7, 29]. A similar statement is valid also for the deformed  $W(D_l)$  [28] and  $U_{q,p}(D_l^{(1)})$ . We also expect that for  $r \in \mathbb{Z}_{>0}$  satisfying  $r > g + 1$  and for a level- $(r - g - 1)$  dominant integral weight  $\mu$ , the space  $\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)$  becomes completely degenerate with respect to the action of the corresponding deformed  $W(\mathfrak{g})$ -algebra [26–28], although the  $E_{6,7,8}$ -type deformed  $W$  algebras have not yet been constructed explicitly. In order to get the irreducible module one should make the BRST-resolution in terms of the BRST-charge constructed from the half currents of  $U_{q,p}(\widehat{\mathfrak{g}})$ . An explicit demonstration for the  $A_1^{(1)}$  case has been discussed in [35].

*Remark.* In Theorem 4.7, we assumed  $p = q^{2r}$  in order to make a connection to the deformed  $W(A_l)$ -algebra. The same relation arises naturally when one considers the finite dimensional representations of the universal elliptic dynamical  $\mathcal{R}$  matrices [5, 40].

### 4.2.2 The $B_l^{(1)}$ case

We follow the work [41] and its quantum analogues [42, 43] with a slight modification in the Ramond sector according to [44]. Let  $e^{\alpha_i}$  ( $i \in I$ ) be the generators of the group algebra  $\mathbb{C}[\mathcal{Q}]$  with the following central extension.

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} e^{\alpha_j} e^{\alpha_i}$$

As before we regard  $h_i$  ( $i \in I$ ) as an operator such that

$$z^{\pm h_i} e^{\alpha_j} = z^{\pm(\alpha_i^\vee, \alpha_j)} e^{\alpha_j} z^{\pm h_i}$$

We also need the Neveu-Schwartz ( $NS$ ) fermion  $\{\Psi_n | n \in \mathbb{Z} + \frac{1}{2}\}$  and the Ramond ( $R$ ) fermion  $\{\Psi_n | n \in \mathbb{Z}\}$  satisfying the following anti-commutation relations.

$$\{\Psi_m, \Psi_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$$

with  $\mathcal{N} = 1/(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$ . We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-\frac{1}{2}}, \Psi_{-\frac{3}{2}} \cdots], \quad \tilde{\mathcal{F}}^R = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \dots]$$

and their submodules  $\mathcal{F}_{even}^{NS,R}$  (reps.  $\mathcal{F}_{odd}^{NS,R}$ ) generated by the even (reps. odd) number of  $\Psi_{-m}$ 's. One should note that for the  $R$  fermion  $\Psi_0^2 = \mathcal{N}$  and  $\{\Psi_m, \Psi_0\} = 0$  for  $m \neq 0$ . So we have two degenerate vacuum states 1 and  $\Psi_0 1$ . We hence consider the extended space

$$\hat{\mathcal{F}}^R = \tilde{\mathcal{F}}^R \otimes \mathbb{C}^2$$

and realize the  $R$ -fermions by

$$\hat{\Psi}_m = \Psi_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \hat{\Psi}_0 = \mathcal{N}^{\frac{1}{2}} (1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that  $\{\hat{\Psi}_m, \hat{\Psi}_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$ . We set

$$\mathcal{F}^R = \mathcal{F}_{even}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}_{odd}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of  $\Psi_m$  on  $\mathcal{F}^{NS}$  is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_m \cdot u = \{\Psi_m, u\} \quad (m \in \mathbb{Z}_{>0}),$$

where  $u \in \mathcal{F}^{NS}$ , whereas  $\hat{\Psi}_m$  acts on  $\mathcal{F}^R$  as

$$\begin{aligned} \hat{\Psi}_{-m} \cdot u \otimes v &= \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \quad \hat{\Psi}_0 \cdot u \otimes v = u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v, \\ \hat{\Psi}_m \cdot u \otimes v &= \{\Psi_m, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \end{aligned}$$

where  $u \in \tilde{\mathcal{F}}^R$ ,  $v \in \mathbb{C}^2$ .

Let us define the fermion fields  $\Psi^{NS}(z)$  and  $\Psi^R(z)$  by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_n z^{-n}, \quad \Psi^R(z) = \sum_{n \in \mathbb{Z}} \hat{\Psi}_n z^{-n}.$$

One can derive the following operator product expansions.

$$\Psi(z)\Psi(w) =: \Psi(z)\Psi(w) : + \langle \Psi(z)\Psi(w) \rangle,$$

where

$$\langle \Psi(z)\Psi(w) \rangle = \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \mathcal{N} \frac{(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases}$$

Then the quantum  $Z$ -algebra  $Z_1(B_l^{(1)})$  is realized as follows [23].

$$\begin{aligned} Z_i^\pm(z) &= e^{\pm\alpha_i} z^{\pm h_i + 1} \quad (1 \leq i \leq l-1), \\ Z_l^\pm(z) &= \frac{1}{\mathcal{N}^{1/2}} \Psi(z) e^{\pm\alpha_l} z^{\pm d_l h_l + d_l}. \end{aligned}$$

There are three irreducible  $Z_1(B_l^{(1)})$ -modules given by

$$\begin{aligned} W(\Lambda_0) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1}, \\ W(\Lambda_1) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1} \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0], \\ W(\Lambda_l) &= \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}] e^{\bar{\Lambda}_l} \cong \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_l} \oplus \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1 + \bar{\Lambda}_l}, \end{aligned}$$

where  $\mathcal{Q}_0$  denotes the sublattice of  $\mathcal{Q}$  generated by the long roots. For generic  $\mu \in \mathfrak{h}^*$  and  $a = 0, 1, l$ , we set  $\mathcal{W}_{H,Q}(\Lambda_a, \mu) = (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ . From Proposition 3.9 we have the following three level-1 irreducible  $U_{q,p}(\hat{B}_l^{(1)})$ -modules with the highest weight  $(\Lambda_a + \mu, \mu)$ :

$$\begin{aligned} \mathcal{V}(\Lambda_a + \mu, \mu) &= \mathcal{F}_{\alpha,1} \otimes_{\mathbb{C}} \mathcal{W}_{H,Q}(\Lambda_a, \mu) \\ &= \bigoplus_{\gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}} \bigoplus_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta}} \mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu), \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^\gamma) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^\gamma) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_l, \gamma, \kappa}(\Lambda_l, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_l - \alpha_l, \gamma, \kappa}(\Lambda_l, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}. \end{aligned}$$

The highest weight vectors are given by  $1 \otimes 1 \otimes 1 \otimes e^{Q_{\bar{\mu}}}$  for  $\mathcal{V}(\Lambda_0 + \mu, \mu)$ ,  $1 \otimes 1 \otimes e^{\bar{\Lambda}_1} \otimes e^{Q_{\bar{\mu}}}$  for  $\mathcal{V}(\Lambda_1 + \mu, \mu)$  and  $1 \otimes 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e^{\bar{\Lambda}_l} \otimes e^{Q_{\bar{\mu}}}$  for  $\mathcal{V}(\Lambda_l + \mu, \mu)$ , respectively.

It is also easy to calculate the characters of these modules:

$$ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} = \text{tr}_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c_W}{24}} = \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } Q_0 + \mathbb{C}\delta \\ \gamma \in \mathfrak{Q}_0, \kappa \in \mathfrak{Q}}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)},$$

where  $c_W = (l + \frac{1}{2}) \left(1 - \frac{2l(2l-1)}{rr^*}\right)$  is the central charge of the  $WB_l$  algebra by Fateev and Lukyanov [31], and the derivation operator  $d$  is realized as

$$\begin{aligned} d &= -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha - N^\Psi + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2) (P + h)^j, \\ N^\alpha &= \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j, \quad N^\Psi = \sum_{m > 0} \frac{m(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}{q^m + q^{-m}} \Psi_{-m} \Psi_m \end{aligned}$$

where  $r, r^* \in \mathbb{C}^\times$ , and  $A_m^j$  are the fundamental weight type elliptic bosons of the type  $B_l$  given in Sec.5.1,  $\Psi_m$  denotes  $\Psi_m$  on  $\mathcal{F}^{NS}$  and  $\hat{\Psi}_m$  on  $\mathcal{F}^R$ . We obtain:

$$\begin{aligned} ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} &= \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } Q_0 + \mathbb{C}\delta \\ \gamma \in \mathfrak{Q}_0, \kappa \in \mathfrak{Q}}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)}, \\ ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu)} &= c_{\Lambda_0}^{\Lambda_0} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\ ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu)} &= c_{\Lambda_1}^{\Lambda_0} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\ ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu)} &= c_{\Lambda_1}^{\Lambda_1} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\ ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu)} &= c_{\Lambda_0}^{\Lambda_1} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\ ch_{\mathcal{F}_{\Lambda_l, \gamma, \kappa}(\Lambda_l, \mu)} &= c_{\Lambda_l}^{\Lambda_l} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_l + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\ ch_{\mathcal{F}_{\Lambda_l - \alpha_l, \gamma, \kappa}(\Lambda_l, \mu)} &= c_{\Lambda_l - \alpha_l}^{\Lambda_l} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_l + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \end{aligned}$$

where

$$\begin{aligned} c_{\Lambda_0}^{\Lambda_0} &= c_{\Lambda_1}^{\Lambda_1} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left( (-q^{\frac{1}{2}}; q)_\infty + (q^{\frac{1}{2}}; q)_\infty \right), \\ c_{\Lambda_1}^{\Lambda_0} &= c_{\Lambda_1}^{\Lambda_0} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left( (-q^{\frac{1}{2}}; q)_\infty - (q^{\frac{1}{2}}; q)_\infty \right), \\ c_{\Lambda_l}^{\Lambda_l} &= c_{\Lambda_l - \alpha_l}^{\Lambda_l} = \frac{q^{\frac{1}{24}}}{2\eta(q)^l} (-q; q)_\infty. \end{aligned}$$

$\sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } Q_0 + \mathbb{C}\delta}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)}$  coincides with the character of the Verma modules of the  $WB_l$ -algebra with the highest weight  $h = \frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \bar{\rho})|^2$  and the central charge  $c_W$  with  $r, r^* = r - 1 \in \mathbb{C}$  being generic.

**Conjecture 4.8.** *There exists a deformation of the  $WB_l$ -algebra such that*

- i) its generating functions commute with the level-1 elliptic currents  $e_j(z)$  and  $f_j(z)$  of  $U_{q,p}(B_l^{(1)})$  modulo a total difference, i.e.  $e_j(z)$  and  $f_j(z)$  at  $c = 1$  are the screening currents of the deformation of the  $WB_l$ -algebra,*
- ii) for generic  $r$  and  $\mu \in \mathfrak{h}^*$ ,  $\mathcal{F}_{\lambda,\xi,\kappa}(\Lambda + \mu, \mu)$  is an irreducible module of the deformation of the  $WB_l$ -algebra.*

*Remark.* All the algebras  $W(\mathfrak{g})$  appearing in sec.4.2.1 and  $WB_l$  in this subsection are the  $W$ -algebras associated with the coset  $X_l^{(1)} \oplus X_l^{(1)} \supset (X_l^{(1)})_{\text{diag}}$  with level  $(r - g - 1, 1)$ . In particular, the  $WB_l$  is different from the one obtained from the quantum Hamiltonian reduction of the affine Lie algebra  $B_l^{(1)}$ . The  $W$ -algebras associated with such coset describe the critical behavior of the face type solvable lattice models introduced by Jimbo, Miwa and Okado [33].

## 5 Elliptic Bosons of Various Types

In this section we introduce elliptic bosons of the fundamental weight type  $A_m^j$  and the orthogonal basis type  $\mathcal{E}_m^{\pm j}$  for  $U_{q,p}(\widehat{\mathfrak{g}})$ ,  $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, C_l^{(1)}, D_l^{(1)}$ . The level-1 bosons  $A_m^j$  and  $\mathcal{E}_m^{\pm j}$  are used to realize the derivation operator  $d$  and the generating function of the deformed  $W(A_l)$ -algebra, respectively, in sec.4.2.

### 5.1 Definition

Let us set  $\eta = -tg/2$  ( $t = (\text{long root})^2/2$ ).

	$A_l^{(1)}$	$B_l^{(1)}$	$C_l^{(1)}$	$D_l^{(1)}$
$g$	$l + 1$	$2l - 1$	$n + 1$	$2l - 2$
$t$	1	1	2	1
$\eta$	$-\frac{l+1}{2}$	$-\frac{2l-1}{2}$	$-(l + 1)$	$-(l - 1)$

Let  $\alpha_{i,m}$  be the elliptic bosons of the simple root type as in Sec.2. We define the fundamental weight type elliptic bosons  $A_m^j$  ( $1 \leq j \leq l, m \in \mathbb{Z}_{\neq 0}$ ) by

$$[\alpha_{i,m}, A_n^j] = -\delta_{i,j} \delta_{m+n,0} \frac{[cm]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} \quad (1 \leq i, j \leq l). \quad (5.1)$$

Note that using the matrix  $B(m) = ([b_{i,j}m])_{1 \leq i,j \leq l}$ , we have [28]

$$A_m^j = \sum_{k=1}^l (B(m)^{-1})_{kj} \alpha_{k,m}.$$

Solving (5.1) we obtain the following.

For  $A_l^{(1)}$ ,

$$A_m^j = C_m \left( [(2\eta + j)m] \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^l [(2\eta + k)m] \alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For  $B_l^{(1)}$ ,

$$A_m^j = C_m \left( (q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^l (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For  $C_l^{(1)}$ ,

$$\begin{aligned} A_m^j &= C_m \left( (q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} \right. \\ &\quad \left. + [jm] \sum_{k=j+1}^{l-1} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} + [jm] \alpha_{l,m} \right), \quad (1 \leq j \leq l-1), \\ A_m^l &= C_m \left( \sum_{k=1}^{l-1} [km] \alpha_{k,m} + \frac{[m]}{[2m]} [lm] \alpha_{l,m} \right). \end{aligned}$$

For  $D_l^{(1)}$ ,

$$\begin{aligned} A_m^j &= C_m \left( (q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} \right. \\ &\quad \left. + [jm] \sum_{k=j+1}^{l-2} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} + [jm] (\alpha_{l-1,m} + a_{l,m}) \right) \quad (1 \leq j \leq l-2), \\ A_m^{l-1} &= C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([lm] \alpha_{l-1,m} + [(l-2)m] a_{l,m}) \right), \\ A_m^l &= C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([lm] \alpha_{l-1,m} + [lm] a_{l,m}) \right). \end{aligned}$$

Here

$$\begin{aligned} C_m &= \frac{1}{[m]^2 [2\eta m]} \quad \text{for } A_l^{(1)} \\ &= \frac{[\eta m]}{[m]^2 [2\eta m]} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}. \end{aligned}$$

We then divide  $A_m^j$  into two terms and define the elliptic bosons  $\mathcal{E}_m^{\pm j}$  of the orthogonal basis type as follows.

For  $A_l^{(1)}$ ,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.2)$$

$$\mathcal{E}_m^{\pm j} = \pm q^{\pm jm} \frac{C_m}{q - q^{-1}} \left( q^{\pm 2\eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} + \sum_{k=j}^l [(2\eta + k)m] \alpha_{k,m} \right) \quad (5.3)$$

for  $1 \leq j \leq l$ . It is convenient to define  $\mathcal{E}_m^{\pm(l+1)}$  by

$$\mathcal{E}_m^{\pm(l+1)} = \mp \frac{C_m}{q - q^{-1}} \sum_{k=1}^l [km] \alpha_{k,m}. \quad (5.4)$$

For  $B_l^{(1)}$ ,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.5)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left( q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^l [(\eta + k)m]_+ \alpha_{k,m} \right) \quad (5.6)$$

for  $1 \leq j \leq l$ . Here we set

$$[m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

We also define

$$\mathcal{E}_m^0 = \frac{\lfloor \frac{m}{2} \rfloor}{[m]} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}). \quad (5.7)$$

For  $C_l^{(1)}$ ,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j} \quad (5.8)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left( q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-1} [(\eta + k)m]_+ \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right) \quad (1 \leq j \leq l-1), \quad (5.9)$$

$$A_m^l = \frac{1}{q^m + q^{-m}} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}), \quad (5.10)$$

$$\mathcal{E}_m^{\pm l} = q^{\pm lm} C_m \left( q^{\pm \eta m} \sum_{k=1}^{l-1} [km] \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right). \quad (5.11)$$

For  $D_l^{(1)}$ ,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.12)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left( q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-2} [(\eta + k)m]_+ \alpha_{k,m} \pm \frac{1}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right) \quad (1 \leq j \leq l-2), \quad (5.13)$$

$$\mathcal{E}_m^{\pm(l-1)} = C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} \pm \frac{q^{\mp \eta m}}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right), \quad (5.14)$$

$$\mathcal{E}_m^{\pm l} = q^{\pm lm} C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} \mp \frac{1}{q - q^{-1}} (q^{\pm \eta m} \alpha_{l-1,m} - q^{\mp \eta m} \alpha_{l,m}) \right). \quad (5.15)$$



**Proposition 5.1.**

$$\alpha_{j,m} = \pm [m]^2 (q - q^{-1}) (\mathcal{E}_m^{\pm j} - q^{\mp m} \mathcal{E}_m^{\pm(j+1)}), \quad (5.16)$$

$1 \leq j \leq l$  for  $A_l^{(1)}$ ,  $1 \leq j \leq l-1$  for  $B_l^{(1)}$ ,  $C_l^{(1)}$ ,  $D_l^{(1)}$ , and

$$\alpha_{l,m} = [m] (q^{m/2} - q^{-m/2}) (q^{-m/2} \mathcal{E}_m^{+l} - q^{m/2} \mathcal{E}_m^{-l}) \quad \text{for } B_l^{(1)}, \quad (5.17)$$

$$= [m]^2 (q - q^{-1}) (q^m \mathcal{E}_m^{+l} - q^{-m} \mathcal{E}_m^{-l}) \quad \text{for } C_l^{(1)}, \quad (5.18)$$

$$= \pm [m]^2 (q - q^{-1}) (\mathcal{E}_m^{\pm(l-1)} - q^{\pm m} \mathcal{E}_m^{\mp l}) \quad \text{for } D_l^{(1)}. \quad (5.19)$$

**Proposition 5.2.** *The following relations hold.*

$$\mathcal{E}_m^{\pm 1} = \pm \frac{q^{\pm m}}{q^m - q^{-m}} A_m^1, \quad \mathcal{E}_m^{\pm j} = \pm \frac{1}{q^m - q^{-m}} (q^{\pm m} A_m^j - A_m^{j-1}), \quad (5.20)$$

where  $2 \leq j \leq l$  for  $A_l^{(1)}$ ,  $2 \leq j \leq l$  for  $B_l^{(1)}$ ,  $2 \leq j \leq l-1$  for  $C_l^{(1)}$  and  $2 \leq j \leq l-2$  for  $D_l^{(1)}$ .

In addition, we have

$$\mathcal{E}_m^{\pm(l+1)} = \mp \frac{1}{q^m - q^{-m}} A_m^l, \quad \sum_{j=1}^{l+1} q^{\pm(j-1)m} \mathcal{E}_m^{\pm j} = 0 \quad \text{for } A_l^{(1)}, \quad (5.21)$$

$$\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} ((q^m + q^{-m}) q^{\pm m} A_m^l - A_m^{l-1}) \quad \text{for } C_l^{(1)}, \quad (5.22)$$

and

$$\mathcal{E}_m^{\pm(l-1)} = \pm \frac{1}{q^m - q^{-m}} (q^{\pm m} A_m^{l-1} + q^{\pm m} A_m^l - A_m^{l-2}), \quad (5.23)$$

$$\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} (q^{\pm 2m} A_m^l - A_m^{l-1}) \quad \text{for } D_l^{(1)}. \quad (5.24)$$

*Remark.* The level-1 case i.e.  $c = 1$ , the  $A_l^{(1)}$  type relation was given in [26, 27] and the  $D_l^{(1)}$  type was essentially given in [28], where parameters  $q$  and  $t$  should be identified with our  $p^{*\frac{1}{2}} = p^{\frac{1}{2}} q^{-1}$  and  $p^{\frac{1}{2}}$ , respectively. However the  $B_l^{(1)}$  and  $C_l^{(1)}$  cases are different from those given in [28]. At least the formulas for  $B_l^{(1)}$  and  $C_l^{(1)}$  seem to be reversed. Our definitions and relations are valid for arbitrary level  $c$ .

Although the expressions of  $\mathcal{E}_m^{\pm j}$  are complicated depending on the types of the affine Lie algebras, their commutation relations are rather universal:

**Theorem 5.3.** *For  $1 \leq j, k \leq l$ , the following commutation relations hold. For  $A_l^{(1)}$ ,*

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \delta_{m+n,0} \frac{[cm][(2\eta+1)m]}{m(q-q^{-1})^2[m]^3[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (5.25)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \delta_{m+n,0} q^{\mp(\text{sgn}(k-j)2\eta+k-j)m} \frac{[cm]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (5.26)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = -\delta_{m+n,0} q^{\pm(2\eta+j+k)m} \frac{[cm]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}. \quad (5.27)$$

For  $B_l^{(1)}, C_l^{(1)}, D_l^{(1)}$ ,

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = \delta_{m+n,0} \frac{[cm][\eta m][2(\eta+1)m]}{m(q-q^{-1})^2[m]^3[2\eta m][(\eta+1)m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (5.28)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \mp \delta_{m+n,0} \frac{q^{\pm jm}[cm][\eta m]}{m[m]^3(q-q^{-1})^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm} \left( q^{\pm(\eta+j)m}[m] \pm q^{\mp(j-1)m}[\eta m]_+ \right), \quad (5.29)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \mp \text{sgn}(k-j) \delta_{m+n,0} q^{\mp(\text{sgn}(k-j)\eta+k-j)m} \frac{[cm][\eta m]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (5.30)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = \mp \delta_{m+n,0} q^{\pm(\eta+j+k)m} \frac{[cm][\eta m]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}. \quad (5.31)$$

Here

$$\text{sgn}(l-j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}$$

*Proof.* Straightforward calculation using Proposition 5.2 and (5.1).  $\square$

**Proposition 5.4.** For  $1 \leq i \leq l$ , the following commutation relations hold.

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm j}] = \pm \delta_{m+n,0} \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1}) \quad (5.32)$$

where  $1 \leq j \leq l$  for  $A_l^{(1)}, B_l^{(1)}$ ,  $1 \leq j \leq l-1$  for  $C_l^{(1)}$ ,  $1 \leq j \leq l-2$  for  $D_l^{(1)}$ . In addition,

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm l}] = \pm \delta_{m+n,0} \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} (q^m + q^{-m}) \delta_{i,l} - \delta_{i,l-1}) \text{ for } C_l^{(1)}, \quad (5.33)$$

and

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm(l-1)}] = \pm \delta_{m+n,0} \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,l-1} + q^{\mp m} \delta_{i,l} - \delta_{i,l-2}), \quad (5.34)$$

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm l}] = \pm \delta_{m+n,0} \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp 2m} \delta_{i,l} - \delta_{i,l-1}) \quad \text{for } D_l^{(1)}. \quad (5.35)$$

From (2.15) and (2.16) we also obtain the following relations.

**Proposition 5.5.** For  $1 \leq j \leq l$ ,

$$[\mathcal{E}_m^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} e_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}), \quad (5.36)$$

$$[\mathcal{E}_m^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}) \quad (5.37)$$

where  $1 \leq i \leq l$  for  $A_l^{(1)}, B_l^{(1)}$ ,  $1 \leq i \leq l-1$  for  $C_l^{(1)}$ ,  $1 \leq i \leq l-2$  for  $D_l^{(1)}$ . In addition,

$$[\mathcal{E}_m^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} e_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}), \quad (5.38)$$

$$[\mathcal{E}_m^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}) \quad \text{for } C_l^{(1)}, \quad (5.39)$$

and

$$[\mathcal{E}_m^{\pm(l-1)}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm m} \delta_{l-1,j} + q^{\pm m} \delta_{l,j} - \delta_{l-2,j}), \quad (5.40)$$

$$[\mathcal{E}_m^{\pm(l-1)}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{l-1,j} + q^{\pm m} \delta_{l,j} - \delta_{l-2,j}), \quad (5.41)$$

$$[\mathcal{E}_m^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm 2m} \delta_{l,j} - \delta_{l-1,j}), \quad (5.42)$$

$$[\mathcal{E}_m^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm 2m} \delta_{l,j} - \delta_{l-1,j}) \quad \text{for } D_l^{(1)}. \quad (5.43)$$

## 5.2 The Elliptic Currents $k_{\pm j}(z)$

Let us set

$$\psi_j(z) =: \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1 - p^m} p^m z^{-m} \right\} :. \quad (5.44)$$

Then the elliptic currents  $\psi_j^{\pm}(z)$  in Definition 2.1 can be written as

$$\psi_j^+(q^{-\frac{c}{2}} z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}} z) = K_j^- \psi_j(pq^{-c} z). \quad (5.45)$$

Let us introduce the new currents  $k_{\pm j}(z)$  ( $1 \leq j \leq l$ ) associated with  $\mathcal{E}_m^{\pm j}$  by

$$k_{\pm j}(z) =: \exp \left\{ \sum_{m \neq 0} \frac{[m]^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} : \quad (5.46)$$

and in addition we define  $k_0(z)$  for  $B_l^{(1)}$  by

$$k_0(z) =: k_{-l}(q^{-1/2} z) \psi_l(q^{-1/2} z) :=: k_{+l}(q^{1/2} z) \psi_l(q^{1/2} z)^{-1} :. \quad (5.47)$$

Then from Proposition 5.1 we have the following decompositions.

### Proposition 5.6.

$$\psi_j(z) =: k_{+j}(z) k_{+(j+1)}(qz)^{-1} :=: k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1} z) : \quad (5.48)$$

where  $1 \leq j \leq l-1$  for  $A_l^{(1)}$ ,  $1 \leq j \leq l-1$  for  $B_l^{(1)}$ ,  $C_l^{(1)}$  and  $D_l^{(1)}$ . In addition,

$$\psi_l(z) =: k_{+l}(z) k_0(q^{-1/2} z)^{-1} :=: k_{-l}(z)^{-1} k_0(q^{1/2} z) : \quad \text{for } B_l^{(1)}, \quad (5.49)$$

$$=: k_{+l}(q^{-1} z) k_{-l}(qz)^{-1} : \quad \text{for } C_l^{(1)}, \quad (5.50)$$

$$=: k_{+(l-1)}(z) k_{-l}(q^{-1} z)^{-1} :=: k_{-(l-1)}(z)^{-1} k_{+l}(qz) : \quad \text{for } D_l^{(1)}. \quad (5.51)$$

Now let us introduce the functions  $\tilde{\rho}^+(z)$ , which appear associated with the elliptic dynamical  $R$ -matrices [40]:

$$\tilde{\rho}^+(z) = \frac{\{q^2 z\} \{\xi^2 q^{-2} z\}}{\{\xi^2 z\} \{z\}} \frac{\{p \xi^2 / z\} \{p / z\}}{\{p \xi^2 q^{-2} / z\} \{p q^2 / z\}} \quad \text{for } A_l^{(1)}, \quad (5.52)$$

$$= \frac{\{\xi z\}^2 \{\xi^2 q^{-2} z\} \{q^2 z\}}{\{\xi^2 z\} \{z\} \{\xi q^2 z\} \{\xi q^{-2} z\}} \frac{\{p \xi^2 / z\} \{p / z\} \{p \xi q^2 / z\} \{p \xi q^{-2} / z\}}{\{p \xi / z\}^2 \{p \xi^2 q^{-2} / z\} \{p q^2 / z\}} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}, \quad (5.53)$$

where  $\xi = q^{-2\eta}$ ,  $\{z\} = (z; p, \xi^2)_\infty$ . The following Theorem indicates a deep relationship between  $k_{\pm j}(z)$ 's and elliptic dynamical  $R$ -matrices.

**Theorem 5.7.**

$$\begin{aligned}
k_{\pm j}(z_1)k_{\pm j}(z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} k_{\pm j}(z_2)k_{\pm j}(z_1), \quad (1 \leq j \leq l), \\
k_{+j}(q^j z_1)k_{+k}(q^k z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)} k_{+k}(q^k z_2)k_{+j}(q^j z_1) \quad (1 \leq j < k \leq l), \\
k_{-j}(q^{-j} z_1)k_{-k}(q^{-k} z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)} k_{-k}(q^{-k} z_2)k_{-j}(q^{-j} z_1) \quad (1 \leq k < j \leq l), \\
k_{+j}(q^j z_1)k_{-k}(q^{-k} \xi z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)} k_{-k}(q^{-k} \xi z_2)k_{+j}(q^j z_1) \quad (j \neq k), \\
k_{+j}(q^j z_1)k_{-j}(q^{-j} \xi z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{2j-2}\xi^{-1}z)\Theta_p(q^{2j}\xi^{-1}z)}{\Theta_{p^*}(q^{2j}\xi^{-1}z)\Theta_p(q^{2j-2}\xi^{-1}z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)} k_{-j}(q^{-j} \xi z_2)k_{+j}(q^j z_1),
\end{aligned}$$

where  $z = z_1/z_2$  and  $\tilde{\rho}^{+*}(z) = \tilde{\rho}^+(z)|_{p \mapsto p^*}$ . In addition, for  $B_l^{(1)}$  we have

$$\begin{aligned}
k_0(z_1)k_0(z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(q^2z)\Theta_{p^*}(qz)\Theta_p(q^{-1}z)}{\Theta_{p^*}(q^2z)\Theta_p(q^{-2}z)\Theta_{p^*}(q^{-1}z)\Theta_p(qz)} k_0(z_2)k_0(z_1), \\
k_{+j}(q^j z_1)k_0(q^{l-1/2} z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)} k_0(q^{l-1/2} z_2)k_{+j}(q^j z_1) \quad (1 \leq j \leq l), \\
k_{-j}(\xi q^{-j} z_1)k_0(q^{l-1/2} z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(z)\Theta_p(q^2z)}{\Theta_{p^*}(q^2z)\Theta_p(z)} k_0(q^{l-1/2} z_2)k_{-j}(\xi q^{-j} z_1) \quad (1 \leq j \leq l).
\end{aligned}$$

*Proof.* Straightforward calculation using Theorem 5.3. □

In addition from Proposition 5.5, we obtain:

**Proposition 5.8.**

$$\begin{aligned}
k_{\pm j}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c\mp 2}z)} e_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l), \\
k_{\pm j}(z_1)e_{j-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)} e_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l), \\
k_{\pm j}(z_1)e_k(z_2) &= e_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1), \\
k_{\pm j}(z_1)f_j(z_2) &= \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)} f_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l), \\
k_{\pm j}(z_1)f_{j-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)} f_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l), \\
k_{\pm j}(z_1)f_k(z_2) &= f_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1)
\end{aligned}$$

for  $A_l^{(1)}, B_l^{(1)}$  with  $1 \leq i \leq l$ ,  $C_l^{(1)}$  with  $1 \leq i \leq l-1$ ,  $D_l^{(1)}$  with  $1 \leq i \leq l-2$ . In addition, we

have

$$\begin{aligned}
k_0(q^{l-1/2}z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c+l}z)\Theta_{p^*}(q^{-c+l-1}z)}{\Theta_{p^*}(q^{-c+l-2}z)\Theta_{p^*}(q^{-c+l+1}z)}e_l(z_2)k_0(q^{l-1/2}z_1), \\
k_0(q^{l-1/2}z_1)e_j(z_2) &= e_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l-1), \\
k_0(q^{l-1/2}z_1)f_l(z_2) &= \frac{\Theta_p(q^{l-2}z)\Theta_p(q^{l+1}z)}{\Theta_p(q^l z)\Theta_p(q^{l-1}z)}f_l(z_2)k_0(q^{l-1/2}z_1), \\
k_0(q^{l-1/2}z_1)f_j(z_2) &= f_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l-1) \quad \text{for } B_l^{(1)}, \\
k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z)}{\Theta_{p^*}(q^{-c\mp 3}z)}e_l(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l-1), \\
k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_p(q^{\mp 3}z)}{\Theta_p(q^{\pm 1}z)}f_l(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l-1) \quad \text{for } C_l^{(1)}, \\
k_{\pm(l-1)}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c\mp 2}z)}e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l-1), \\
k_{\pm(l-1)}(z_1)e_{l-2}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
k_{\pm(l-1)}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l-1, l-2), \\
k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\mp 3}z)}e_l(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l-1), \\
k_{\pm(l-1)}(z_1)f_j(z_2) &= \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)}f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l-1), \\
k_{\pm(l-1)}(z_1)f_{l-2}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
k_{\pm(l-1)}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l-1, l-2), \\
k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_p(q^{\mp 3}z)}{\Theta_p(q^{\mp 1}z)}f_l(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l-1) \quad \text{for } D_l^{(1)}.
\end{aligned}$$

The elliptic bosons  $\mathcal{E}_m^{\pm j}$  and their elliptic currents  $k_{\pm j}(z)$  are useful to realize the  $L$ -operators and the vertex operators for  $U_{q,p}(\widehat{\mathfrak{g}})$  as well as deformation of the  $W$ -algebras. We will discuss this subject in separate papers.

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## A The Drinfeld Realization of $U_q(\widehat{\mathfrak{g}})$

Let  $\widehat{\mathfrak{g}}$  be an untwisted affine Lie algebra.

**Definition A.1.** *The quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  in the Drinfeld realization is a unital  $\mathbb{C}$ -algebra generated by  $q^h$  ( $h \in \mathfrak{h}$ ),  $a_{i,n}^\vee$ ,  $x_{i,m}^\pm$  ( $i \in I$ ,  $n \in \mathbb{Z}_{\neq 0}, m \in \mathbb{Z}$ )  $\bar{d}$  and the central element  $c$ . We set*

$$x_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^{-m}, \quad (\text{A.1})$$

$$\psi_i(z) = q_i^{h_i} \exp \left( (q_i - q_i^{-1}) \sum_{n>0} a_{i,n}^\vee z^{-n} \right), \quad (\text{A.2})$$

$$\varphi_i(z) = q_i^{-h_i} \exp \left( -(q_i - q_i^{-1}) \sum_{n>0} a_{i,-n}^\vee z^n \right). \quad (\text{A.3})$$

The defining relations are as follows.

$$[q_i^{\pm h_i}, \bar{d}] = 0, \quad [\bar{d}, a_{i,n}] = n a_{i,n}, \quad [\bar{d}, x_{i,n}^\pm] = n x_{i,n}^\pm, \quad (\text{A.4})$$

$$[q_i^{\pm h_i}, a_{j,n}] = 0, \quad q_i^{h_i} x_j^\pm(z) = q_i^{\pm a_{ij}} x_j^\pm(z) q_i^{h_i}, \quad (\text{A.5})$$

$$[a_{i,n}^\vee, a_{j,m}^\vee] = \frac{[a_{ij}n]_i [cn]_j}{n} q^{-c|n|} \delta_{n+m,0}, \quad (\text{A.6})$$

$$[a_{i,n}^\vee, x_j^+(z)] = \frac{[a_{ij}n]_i}{n} q^{-c|n|} z^n x_j^+(z), \quad (\text{A.7})$$

$$[a_{i,n}^\vee, x_j^-(z)] = -\frac{[a_{ij}n]_i}{n} z^n x_j^-(z), \quad (\text{A.8})$$

$$(z - q^{\pm b_{ij}} w) x_i^\pm(z) x_j^\pm(w) = (q^{\pm b_{ij}} z - w) x_j^\pm(w) x_i^\pm(z), \quad (\text{A.9})$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left( \delta(q^{-k} \frac{z}{w}) \psi_i(q^{k/2} w) - \delta(q^k \frac{z}{w}) \varphi_i(q^{-k/2} w) \right), \quad (\text{A.10})$$

$$\sum_{\sigma \in S_a} \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(s)}) x_j^\pm(w) x_i^\pm(z_{\sigma(s+l)}) \cdots x_i^\pm(z_{\sigma(a)}) = 0, \quad (\text{A.11})$$

( $i \neq j$ ,  $a = 1 - a_{ij}$ ).

For  $k \in \mathbb{C}$ , we define the category  $C_k$  of the level- $k$   $U_q(\widehat{\mathfrak{g}})$ -modules in the same way as  $\mathfrak{C}_k$  of  $U_{q,p}(\widehat{\mathfrak{g}})$  in Sec.2. Let  $a_{i,n} = [d_i] a_{i,n}^\vee$  ( $i \in I, n \in \mathbb{Z}_{\neq 0}$ ) be the simple root type level- $k$  Drinfeld bosons. They satisfy

$$[a_{i,n}, a_{j,m}] = \frac{[b_{ij}n][kn]}{n} q^{-k|n|} \delta_{n+m,0}.$$

For  $(V, \bar{\pi}) \in C_k$ , we define the  $Z$ -operators associated with the level- $k$   $U_q(\widehat{\mathfrak{g}})$ -module  $V$  by

$$Z_i^\pm(z; V) = \exp \left( \mp \sum_{n \geq 1} \frac{\bar{\pi}(a_{i,-n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^n \right) \bar{\pi}(x_i^\pm(z)) \exp \left( \pm \sum_{n \geq 1} \frac{\bar{\pi}(a_{i,n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^{-n} \right).$$

The coefficients  $Z_{i,n}^\pm(V)$  of  $Z_i^\pm(z; V) = \sum_{n \in \mathbb{Z}} Z_{i,n}^\pm(V) z^{-n}$  in  $z$  are well defined elements in  $\text{End}_{\mathbb{C}} V$ .

**Theorem A.2.** *The  $Z$ -operators  $Z_i^\pm(z; V)$  satisfy the same relations in Theorem 3.3 except for (3.16),(3.17) with replacement  $\mathcal{Z}_j^\pm(z; \mathcal{V})$ ,  $\alpha_{j,m}$ ,  $d$  and  $K_j^\pm$  by  $Z_i^\pm(z; V)$ ,  $a_{j,m}$ ,  $\bar{d}$  and  $q_j^{\mp h_j}$ , respectively.*

*Remark.* This theorem is essentially due to Jing [23]. However, in [23] no Serre relations are written explicitly. There are also some misprints in Theorem 2.2 in [23]:

- $(1 - q^\mp w/z)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$  should be read as  $(1 - q^\mp w/z)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$
- $(1 - q^\mp z/w)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$  should be read as  $(1 - q^\mp z/w)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$
- $(1 - w/z)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$  should be read as  $(1 - w/z)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$
- $(1 - z/w)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$  should be read as  $(1 - z/w)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$

**Definition A.3.** *For  $k \in \mathbb{C}^\times$  and  $(V, \bar{\pi}) \in C_k$ , we call the subalgebra of  $\text{End}_{\mathbb{C}} V$  generated by  $Z_{i,m}^\pm(V)$ ,  $q_i^{\pm h_i}$  ( $i \in I, m \in \mathbb{Z}$ ) and  $\bar{d}$  the quantum  $Z$ -algebra  $Z_V$  associated with  $(V, \bar{\pi})$ . We also define the universal quantum  $Z$  algebra  $Z_k$  as a topological algebra over  $\mathbb{C}[[q^{2k}]]$  in the same way as  $\mathcal{Z}_k$  in Definition 3.5. We denote the generators in  $Z_k$  by  $Z_{j,m}^\pm$  ( $j \in I$ ).*

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